TECHNISCHE UNIVERSITEIT EINDHOVEN
Faculteit Wiskunde en Informatica

Exam Stochastische Processen 2 (2S480) on 19 November 2003, 9.00-12.00 hours.

Please write the answers and reasonings with a clear formulation.

1. Consider a Markov process $\{X(t), t \geq 0\}$ with four states (A, B, C, D) and with the following Q-matrix:

$$\begin{pmatrix} -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -2 & 1 & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -2 & 1 \\ \frac{1}{3} & 1 & \frac{2}{3} & -2 \end{pmatrix} \tag{1}$$

a. [3 pt.] Determine the equilibrium distribution (i.e., the limiting distribution) of the Markov process (Hint: argue that the equilibrium probabilities for the states B, C and D are equal).
b. [2 pt.] Determine the matrix P of one-step transition probabilities of the Markov chain that belongs to this Markov process (at jump epochs).
c. [3 pt.] Determine the equilibrium distribution of this Markov chain, by using the equilibrium distribution for the Markov process found under a).
d. [2 pt.] Explain how $P(X(t) = A|X(0) = A)$ can be obtained with the help of the uniformization method (I don't ask for a calculation).

2. Consider the birth-and-death process $\{X(t), t \geq 0\}$ on the state space $\{0, 1, \ldots, n\}$, where $X(0) = 0$. The birth rate is λ and the death rate is μ, with the following exception: when state n is reached, the process does not change anymore.
a. [2 pt.] Give the Q-matrix of this process.
b. [3 pt.] Give the forward differential equations of Kolmogorov for the process, for the probabilities $P(X(t + s) = i|X(s) = i)$, $i = 0, \ldots, n$.
c. [3 pt.] Let $n = 1$. Determine $P(X(t) = 0|X(0) = 0)$ and $P(X(t) = 1|X(0) = 0)$.
d. [2 pt.] Let $n = 2$. Determine the expected time until state 2 is reached.

3. Consider a branching process $\{X_n, n = 0, 1, \ldots\}$, starting with one particle: $X_0 = 1$. The number of offspring Z of one particle equals i with probability p_i, where $p_0 = c$, $p_1 = \frac{1}{6}$, $p_2 = \frac{5}{6} - c$.
a. [3 pt.] For which values of c is extinction of the population certain?
b. [3 pt.] Determine $E(X_n)$, $n = 1, 2, \ldots$.
c. [4 pt.] Same questions if the number of offspring per particle in the even generations is as above (hence $P(X_1 = i) = p_i$, $i = 0, 1, 2$), whereas in the odd generations, the number of offspring of a particle is Poisson distributed with mean 2 (Hint: observe that $\{X_{2n}, n = 0, 1, \ldots\}$ again is an ordinary branching process).

4. Consider the renewal process $\{N(t), t \geq 0\}$, with distribution $F(\cdot)$ and density $f(\cdot)$ of the intervals between successive renewals.
a. [3 pt.] Argue that $m(t) := E(N(t))$ satisfies the following equation:

$$m(t) = F(t) + \int_0^t m(t-x) f(x) \mathrm{d}x, \quad t \geq 0.$$

b. [4 pt.] In the remainder of the exercise, take $f(t) = \lambda^k \frac{t^{k-1}}{(k-1)!} \mathrm{e}^{-\lambda t}$, i.e., the Erlang-$k$ density. Determine the Laplace-Stieltjes transform of the renewal function $m(t)$ and of $[E(S_{N(t)+1}) - t]$.
c. [3 pt.] What is $\lim_{t \to \infty} \frac{m(t)}{t}$?