TECHNISCHE UNIVERSITEIT EINDHOVEN
Faculteit Wiskunde en Informatica

Tentamen Stochastische Processen 2 (2S480) op 2 juli 2002, 09.00-12.00 uur.

1. Consider a Markov process with three states (A, B, C) and with the following Q-matrix:

$$\begin{pmatrix} -4 & 2 & 2 \\ 3 & -5 & 2 \\ 3 & 1 & -4 \end{pmatrix} \tag{1}$$

a) [2 pt.] Determine the matrix P of one-step transition probabilities of the Markov chain belonging to this Markov process.
b) [3 pt.] Determine the equilibrium distribution (i.e., limiting distribution) of the Markov process.
c) [3 pt.] What is the relation between the equilibrium distributions of the Markov chain and the Markov process?
d) [2 pt.] SKIP PART d).

2. Consider a factory with K machines. A machine works for an exponentially distributed time, with mean $1/\lambda$, and then breaks down. There is one repairman, who repairs broken machines in order of breakdown. The repair times are exponentially distributed with mean $1/\mu$. All mentioned times are independent. A repaired machine immediately goes back into operation. Let $\{X_t,\ t \geq 0\}$ be the number of operational machines at time t.
a) [2 pt.] Argue that $\{X_t,\ t \geq 0\}$ is a Markov process.
b) [2 pt.] Give the Q-matrix of transition rates of this Markov process.
c) [2 pt.] For which values of λ and μ does the equilibrium distribution of this Markov process exist?
d) [2 pt.] Give the equilibrium distribution in case $\mu = \lambda$.
e) [4 pt.] Let $K = 1$ and $X(0) = 0$. Give the forward differential equations of Kolmogorov for the Markov process $\{X_t,\ t \geq 0\}$, and solve these equations.

3. Consider the $E_2/M/2/4$ queue. This is a system with Erlang-2 distributed arrival intervals, 2 servers and a waiting room of size 2. Service times are exponentially distributed.
a) [3 pt.] We are interested in the number of customers in the system. Define a suitable Markov process for this system and draw the corresponding state-transition diagram.
b) [3 pt.] Give all (global) balance equations.

4. Consider the following branching process. Let X_n be the number of particles in generation n; $X_0 = 1$. Each particle, independently of other particles, gives birth to j particles with probability p_j, $j = 0, 1, \ldots$. The generating function of this distribution is given by $P(z) = \sum_{j=0}^{\infty} p_j z^j = \mathrm{e}^{-\lambda(1-z)} \frac{1-c}{1-cz}$, with $0 < \lambda < 1/2$ en $0 < c < 1$.
a) [2 pt.] What is the expected number of descendants of a particle, and what is the probability that a particle has exactly one descendant?

b) [2 pt.] What is $E(X_n)$?
c) [3 pt.] For which values of c (as function of λ) will the population extinguish with probability one?
d) [2 pt.] Consider two independent branching processes $\{X_{1,n}, n = 0, 1, \ldots\}$ and $\{X_{2,n}, n = 0, 1, \ldots\}$. $X_{1,0} = X_{2,0} = 1$. The generating function of the number of children of a particle in the first branching process is $e^{-\lambda(1-z)}$, and for the second process it is $\frac{1-c}{1-cz}$. What is the probability that after two generations both populations are no longer existent?
e) [3 pt.] What is the sum of the expected numbers of particles in the n-th generation of the two branching processes; explain the fact that this result is for $n = 1$ the same as in question b).