A linear programming formulation of Mader's edge-disjoint paths problem

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Abstract

We give a dual pair of linear programs for a min-max result of Mader describing the maximum number of edge-disjoint T-paths in a graph $G = (V, E)$ with $T \subseteq V$. We conclude that there exists a polynomial-time algorithm (based on the ellipsoid method) for finding a maximum number of T-paths in a capacitated graph, where the number of T-paths using an edge does not exceed the capacity of that edge.

Keywords: disjoint paths, path packing, ellipsoid method

1 Introduction

Let $G = (V, E)$ be a graph and $T \subseteq V$ a set of *terminals.* A *T-path* is a path in G connecting different vertices of T. Let $\mu(G, T)$ denote the maximum number of edge-disjoint T-paths in G.

A *T-subpartition* in G is a set $\mathcal{P}$ of $|T|$ disjoint subsets of V, each containing exactly one vertex from T. If $\mathcal{P}$ is a T-subpartition in G then we denote the set of connected components of $G \setminus \cup\mathcal{P}$ by $\mathcal{C}(\mathcal{P})$. For two disjoint sets $U_1, U_2 \subseteq V$, we denote by $\delta_G(U_1, U_2)$ the set of edges from G with one endpoint in U_1 and the other in U_2. We write $d_G(U_1, U_2)$ for $|\delta_G(U_1, U_2)|$ and $\delta_G(U)$, $d_G(U)$ for $\delta_G(U, V \setminus U)$ and $d_G(U, V \setminus U)$ respectively. If the graph G is clear from the context, the subscripts G are left out. If $\mathcal{P}$ is a T-subpartition in G, then the number of components U in $\mathcal{C}(\mathcal{P})$ with $d_G(U)$ odd (these will be referred to as *odd components* of $\mathcal{P}$) is denoted $q(\mathcal{P})$. It is clear that

$$\mu(G, T) \leq 1/2 \min\{(\sum_{P \in \mathcal{P}} d_G(P)) - q(\mathcal{P}) \mid \mathcal{P} \text{ a } T\text{-subpartition of } G\},$$

since every T-path has to leave one element and enter another element of the T-subpartition $\mathcal{P}$, whereas for every odd component of $\mathcal{P}$ there is at least one edge entering it (this edge has its other end in an element of $\mathcal{P}$) that is not used in any path of a collection of edge-disjoint T-paths. Define

$$\nu(G, T) := 1/2 \min\{(\sum_{P \in \mathcal{P}} d_G(P)) - q(\mathcal{P}) \mid \mathcal{P} \text{ a } T\text{-subpartition of } G\}.$$

Note that $\nu(G, T)$ is equal to $\sum_{P,Q \in \mathcal{P}} d_G(P, Q) + \sum_{U \in \mathcal{C}(\mathcal{P})} \lfloor d_G(U)/2 \rfloor$. Mader [4] proved the following min-max relation for the maximum number of T-paths in a graph $G = (V, E)$.

Theorem 1.1 (Mader) *For a graph $G = (V, E)$ and set of terminals $T \subseteq V$,*

$$\mu(G, T) = \nu(G, T).$$

In this note, we will show that the above min-max relation can be interpreted as strong duality for two dual linear programs having integer optimal solutions. We will show that it is possible to separate over the feasible region of the primal problem in polynomial time. This implies that there exists a polynomial time algorithm (using the ellipsoid method) for finding a maximum number of edge-disjoint T-paths in a graph with a capacity function $c : E \to \mathbb{Z}_+$, such that at most $c(e)$ of the paths use edge e for every $e \in E$ (the min-max result remains valid for this case, since one can simply replace an edge e with capacity $c(e)$ by $c(e)$ parallel edges). The question whether a combinatorial polynomial time algorithm for this problem exists remains open. The uncapacitated case however is a special case of linear matroid matching, cf. [2], and therefore it can be solved by a combinatorial algorithm, cf. [3].

2 Linear programming formulation

Let $T \subseteq V$ be a set of terminals in the graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$. Let $\mathcal{T}$ denote the set of $1/2|T|(|T|-1)$ ordered pairs $\{(u, v) \mid u, v \in T,\ u < v\}$, and let E^* denote the set of $2|E|$ ordered pairs $\{(i, j) \mid \{i, j\} \in E\}$. To encode a collection of edge-disjoint T-paths we will use binary vectors $x = (x_{ij}^{uv})_{(i,j)\in E^*}^{(u,v)\in\mathcal{T}}$ of dimension $|E||T|(|T|-1)$, where $x_{ij}^{uv} = 1$ indicates that in the collection there is a T-path between $u, v \in T$ with $u < v$, which, when it is oriented from u to v, uses edge $\{i, j\} \in E$ in the direction from i to j. Now, it is not difficult to see that $\mu(G, T)$ is at most equal to the optimal value of the following integer program.

$$\max \quad 1/2 \sum_{(u,v)\in\mathcal{T}} \sum_{j\in V} (x_{uj}^{uv} - x_{ju}^{uv} + x_{jv}^{uv} - x_{vj}^{uv}) \quad \text{s.t.}$$

$$\sum_{(u,v)\in\mathcal{T}} \sum_{j\in V} (x_{ij}^{uv} - x_{ji}^{uv}) = \sum_{(v,u)\in\mathcal{T}} \sum_{j\in V} (x_{ij}^{vu} - x_{ji}^{vu}) \quad \forall u \in T, \forall i \in V \setminus T$$

$$\sum_{(u,v)\in\mathcal{T}} (x_{ij}^{uv} + x_{ji}^{uv}) \le 1 \quad \forall \{i, j\} \in E$$

$$\sum_{(u,v)\in\mathcal{T}} \sum_{\{i,j\}\in\delta(X)} (x_{ij}^{uv} + x_{ji}^{uv}) \le d(X) - 1 \quad \forall X \subseteq V \setminus T \text{ with } d(X) \text{ odd}$$

$$x_{ij}^{uv} \in \{0, 1\} \quad \forall (u, v) \in \mathcal{T}, \forall (i, j) \in E^*$$

In the above integer program, the equality constraints can be viewed as flow conservation constraints: they say that for every terminal u and every non-terminal vertex i, there are equally many u–$(T \setminus \{u\})$ paths entering i and leaving i in a system of T-paths. The constraints of the second type are capacity constraints: they say that paths in a system of T-paths are edge-disjoint. The constraints of the third type are the odd-set constraints: they say that at least one edge from every odd cut $\delta(X)$, with $X \subseteq V \setminus X$, is not used by any path in a system of edge-disjoint T-paths. These constraints are superfluous in the integer program, but will imply integrality for the linear programming relaxation.

Next, we consider the linear programming relaxation of the above integer program: we allow the variables x_{ij}^{uv} to take values in $\mathbb{R}_+$ rather than just in $\{0,1\}$. In addition, we introduce nonnegative slack variables y_e for every $e = \{i,j\} \in E$ (corresponding to the capacity constraints). Now $y_e = 1$ means that edge e is not used (in either direction) by a feasible solution. The resulting linear program can be rewritten as follows.

$$\max \quad 1/2 \sum_{u \in T} z_u^u \quad \text{s.t.}$$

$$\sum_{(u,v) \in \mathcal{T}} \sum_{j \in V} (x_{ij}^{uv} - x_{ji}^{uv}) - \sum_{(v,u) \in \mathcal{T}} \sum_{j \in V} (x_{ij}^{vu} - x_{ji}^{vu}) = z_i^u \quad \forall u \in T, \forall i \in V$$

$$z_i^u = 0 \quad \forall u \in T, \forall i \in V \setminus T$$

$$\sum_{(u,v) \in \mathcal{T}} (x_{ij}^{uv} + x_{ji}^{uv}) + y_e = 1 \quad \forall e = \{i,j\} \in E$$

$$\sum_{e \in \delta(X)} y_e \geq 1 \quad \forall X \subseteq V \setminus T \text{ with } d(X) \text{ odd}$$

$$x_{ij}^{uv}, x_{ji}^{uv}, y_e \geq 0 \quad \forall (u,v) \in \mathcal{T}, \forall e = \{i,j\} \in E$$

$$z_i^u \in \mathbb{R} \quad \forall u \in T, \forall i \in V.$$

The dual of the above linear program is the following.

$$\min \quad \sum_{e \in E} \lambda_e - \sum_{X \subseteq V \setminus T \text{ odd}} \sigma_X \quad \text{s.t.}$$

$$\lambda_e + \pi_{ui} - \pi_{uj} - \pi_{vi} + \pi_{vj} \geq 0 \quad \forall (u,v) \in \mathcal{T}, \forall e = \{i,j\} \in E$$

$$\pi_{uu} = -1/2 \quad \forall u \in T$$

$$\pi_{uv} = 0 \quad \forall u, v \in T, u \neq v$$

$$\lambda_e - \sum_{X \subseteq V \setminus T \text{ odd: } e \in \delta(X)} \sigma_X \geq 0 \quad \forall e = \{i,j\} \in E$$

$$\sigma_X \geq 0 \quad \forall X \subseteq V \setminus T \text{ odd}$$

$$\lambda_e, \pi_{ui} \in \mathbb{R} \quad \forall e \in E, \forall u \in T, \forall i \in V$$

It can be shown that the optimal value $\nu(G,T)$ of a Mader-dual is greater or equal to the optimal value of the above dual linear program. Indeed, if we are given an optimal T-partition $\mathcal{P}$, where the element of $\mathcal{P}$ containing $u \in T$ is denoted P_u, then a feasible solution for the dual linear program can be defined from it as follows.

$$
\begin{array}{lcll}
\pi_{uu} & = & -1/2 & \forall u \in T \\
\pi_{uv} & = & 0 & \forall u, v \in T,\ u \neq v \\
\pi_{ui} & = & -1/2 & \forall i \in P_u \setminus \{u\} \\
\pi_{ui} & = & 0 & \forall i \in V \setminus P_u \\
\sigma_X & = & 1/2 & \forall X \in \mathcal{C}(\mathcal{P}) \text{ odd} \\
\sigma_X & = & 0 & \text{ otherwise} \\
\lambda_e & = & 1 & \forall e = \{i, j\} \in \delta(P_1, P_2),\ P_1, P_2 \in \mathcal{P} \text{ with } P_1 \neq P_2 \\
\lambda_e & = & 1/2 & \forall e = \{i, j\} \in \delta(P, X),\ P \in \mathcal{P},\ X \in \mathcal{C}(\mathcal{P}) \\
\lambda_e & = & 0 & \text{ otherwise}
\end{array}
$$

With these choices for the values of the variables, the value of the dual program is equal to the value $\nu(G, T)$ of the given optimal T-partition. Because $\mu(G, T)$ is obviously smaller than or equal to the optimal value of the primal program, Mader's theorem proves that the above linear programs have integer optimal solutions, and that their optimal value is $\mu(G, T)$.

Now, to see that it is possible to separate in polynomial time over the feasible region of the primal linear program, it suffices to prove that one can separate in polynomial time over the polyhedron

$$\{y \in \mathbb{R}_+^E \mid y(\delta(X)) \geq 1\, \forall X \subseteq V \setminus T \text{ with } d(X) \text{ odd } \}.$$

Separating over this polyhedron amounts to finding an odd cut of minimum capacity in a capacitated graph, which can be done in strongly polynomial time ([5]). By the ellipsoid method (cf [1]), it is also possible to optimize over the feasible region of the primal linear program in polynomial time, and hence to determine $\mu(G, T)$ for a given graph G in polynomial time. Consequently, an optimal system of edge-disjoint T-paths in G can be found in polynomial time (for example by iteratively deleting edges as long as this does not reduce the μ-value of the graph).

After seeing the above LP-formulation, both A. Schrijver [6, page 1285] and A. Sebő [7] (independently) suggested a more concise LP-formulation for Mader's edge disjoint T-paths problem, which is worth mentioning.

$$\mu(G,T) = \max \quad 1/2 \sum_{t \in T} x(\delta(t)) \quad s.t.$$

$$0 \le x_e \le 1 \quad \forall e \in E$$

$$x(\delta(X)) \le d(X) - 1 \quad \forall X \subseteq V \setminus T \text{ with } d(X) \text{ odd}$$

$$x(\delta(t)) \le x(\delta(X)) \quad \forall t \in T, \forall X \subseteq V \text{ with } X \cap T = \{t\}$$

A short proof of this identity, and of the fact that it is possible to separate in polynomial time over the feasible region of this linear program, appears in [6, page 1285].

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