A note on packing connectors

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Abstract

Given an undirected graph $G = (V, E)$ and a partition $\{S, T\}$ of V, an S–T *connector* is a set of edges $F \subseteq E$ such that every component of the subgraph (V, F) intersects both S and T.

We show that G has k edge-disjoint S–T connectors if and only if $|\delta_G(V_1) \cup \ldots \cup \delta_G(V_t)| \geq kt$ for every collection $\{V_1, \ldots, V_t\}$ of disjoint nonempty subsets of S and for every such collection of subsets of T.

This is a common generalization of a theorem of Tutte and Nash-Williams on disjoint spanning trees and a theorem of König on disjoint edge covers in a bipartite graph.

1 Introduction

Let $G = (V, E)$ be an undirected graph, S a subset of its vertices, and T the complement of S in V. An S–T *connector* in G is a set F of edges such that every component of the subgraph (V, F) intersects both S and T. Let k be a nonnegative integer. In this note, we prove the following theorem on packing S–T connectors.

Theorem 1 *G contains k edge-disjoint S–T connectors if and only if $|\delta(W)| \geq k|W|$ for every subpartition W of S or T.*

A *subpartition* W of a set X is a collection of pairwise disjoint nonempty subsets of X. If $W = \{U_1, \ldots, U_t\}$ is a subpartition of S or T, then $\delta(W)$ denotes the set of edges with one end in U_i and one end in $V \setminus U_i$ for some index i.

Theorem 1 has two well-known special cases. First, if G is bipartite with colour classes S and T, then an S–T connector is an edge cover of G (a set of edges covering all vertices), and Theorem 1 specializes to a theorem of König

[5] and Gupta [2], saying that the maximum number of edge-disjoint edge covers of a bipartite graph is equal to the minimum vertex degree. Second, if either S or T is a singleton, then an S–T connector is a connected spanning subgraph of G, and Theorem 1 specializes to a result of Tutte [9] and Nash-Williams [6], giving a necessary and sufficient condition for a graph to have k disjoint spanning trees. We state this result here as a lemma, since we will use it in the proof of Theorem 1.

Lemma 1 *Let $G = (V, E)$ be an undirected graph. Then G contains k edge-disjoint spanning trees if and only if $|\delta(P)| \geq k(|P| - 1)$ for every partition P of V into nonempty subsets.* □

Lemma 1 is a special case of the matroid base packing theorem.

At this point, observe that an S–T connector is a common spanning set of two matroids on E, namely the cycle matroids of the graphs G_S and G_T, respectively. Here, G_S is the graph obtained from G by shrinking the set S into a single vertex s (if an edge of G connects two vertices in S, then in G_S there is a loop corresponding to this edge), and G_T, t are defined similarly. Therefore, matroid intersection provides a min-max relation for the minimum cardinality (or weight) of an S–T connector in G. However, no general theorem is known for the packing of common spanning sets of two matroids. Thus, our theorem gives a case where a min-max relation for packing common spanning sets of two matroids *is* possible (although graphic matroids generally are not "strongly base orderable"). (For matroid theory we refer to [10].)

The concept of an S–T connector in an undirected graph is related to the concept of a bibranching in a directed graph. Given a directed graph $D = (V, A)$ and a set $S \subseteq V$ (with $T := V \setminus S$), an S–T *bibranching* is a set of arcs $B \subseteq A$ containing a directed v–T path for every $v \in S$ and a directed S–v path for every $v \in T$.

With respect to packing bibranchings, Schrijver [7] proved the following result, which is the second constituent of the proof of Theorem 1.

Lemma 2 *Let $D = (V, A)$ be a digraph, let $S \subset V$, and let $T = V \setminus S$. Then D contains k arc-disjoint S–T bibranchings if and only if $|\delta_D^+(U)| \geq k$ for every nonempty $U \subseteq S$ and $|\delta_D^-(U)| \geq k$ for every nonempty $U \subseteq T$.* □

Here, $\delta_D^+(U)$ denotes the set of arcs leaving U and $\delta_D^-(U)$ denotes the set of arcs entering U in D.

2 Packing connectors

In this section, we prove Theorem 1, by combining Lemma 1 and Lemma 2.

Proof of Theorem 1.
Necessity is straightforward. To see sufficiency, let G be such that $|\delta(W)| \geq k|W|$ for every subpartition W of S or T. Then G_S satisfies the condition of Lemma 1 (if P is a partition of the vertex set of G_S, omit the class of P that contains s to obtain a subpartition W of T with $|\delta(W)| = |\delta(P)|$ and $|W| = |P| - 1$). Therefore, it contains k disjoint spanning trees. The same holds for G_T. Now orient the edges of the spanning trees in G_S away from s and orient the edges of the spanning trees in G_T towards t. Note that there is no conflict for edges that are both in a spanning tree of G_S and in a spanning tree of G_T, since these edges connect S and T. Orienting the remaining edges of G arbitrarily, we obtain an orientation D of G. Clearly, $|\delta_D^-(U)| \geq k$ for every $U \subseteq T$ and $|\delta_D^+(U)| \geq k$ for every $U \subseteq S$. Therefore, by Lemma 2, D contains k arc-disjoint S–T bibranchings. Since each bibranching in D gives an S–T connector in G, this implies the theorem. $\square$

The above proof gives rise to a polynomial algorithm for packing S–T connectors. Indeed, packing spanning trees can be done with any matroid partition algorithm (or alternatively, Barahona [1] reduces the problem to maximum flow computations). Moreover, disjoint bibranchings can be found in polynomial time, using the ellipsoid method (see [7]). A direct combinatorial algorithm for packing connectors is described in a subsequent paper [3]. An extension of the method used in that paper also yields a combinatorial algorithm for packing bibranchings.

For the problem of finding a minimum-weight bibranching, a combinatorial algorithm is described in [4].

3 Polyhedral interpretation

In this section we show that Theorem 1 implies the integer rounding property for a set of linear inequalities associated with packing S–T connectors. (For background, see [8].)

Assume that G contains an S–T connector. Equivalently, both G_S and G_T are connected. Because an S-T-connector is a common spanning set of two matroids, the convex hull of all incidence vectors of S–T connectors in

G can be derived from the theory of matroid polytopes:

$$\begin{aligned}&\text{conv.hull}\{\chi^F \mid F \in \mathcal{F}\}\\ &\quad= \{x \in \mathbb{R}^E \mid 0 \le x \le 1,\ x(\delta(W)) \ge |W| \text{ for each } W \in \mathcal{W}\}.\end{aligned}$$

Here, χ^F denotes the incidence vector of a set $F \subseteq E$, and $\mathcal{F}$ the set of all S–T connectors of G. Moreover, $\mathcal{W}$ denotes the set of all subpartitions of S and T. Finally, if $x \in \mathbb{R}^E$ and $F \subseteq E$, $x(F)$ is short for $\sum_{e\in F} x(e)$.

It follows that the polyhedra

$$P := \text{conv.hull}\{\chi^F \mid F \in \mathcal{F}\} + \mathbb{R}^E_+$$

and

$$Q := \text{conv.hull}\{\chi^{\delta(W)}/|W| \mid W \in \mathcal{W}\} + \mathbb{R}^E_+$$

form a blocking pair. In other words, $P = \{z \in \mathbb{R}^E_+ \mid x^T z \ge 1\ \forall x \in Q\}$ and $Q = \{x \in \mathbb{R}^E_+ \mid z^T x \ge 1\ \forall z \in P\}$.

Now, let M be the $\mathcal{F} \times E$ matrix with rows the incidence vectors of all S-T connectors of G. Then the fact that P and Q form a blocking pair implies:

$$\text{(1)}\quad \begin{aligned}\min\{w^T\chi^{\delta(W)}/|W| \mid W \in \mathcal{W}\} &= \min\{w^T x \mid x \ge 0, Mx \ge \mathbf{1}\}\\ &= \max\{y^T\mathbf{1} \mid y \ge 0, y^T M \le w\}.\end{aligned}$$

The last equality is linear programming duality.

Theorem 1 has the following polyhedral formulation:

Theorem 2 *For every* $w : E \to \mathbb{Z}_+$

$$\begin{aligned}&\max\{y^T\mathbf{1} \mid y \ge 0, y^T M \le w, y \textit{ integral}\}\\ &\quad= \lfloor \min\{w^T\chi^{\delta(W)}/|W| \mid W \in \mathcal{W}\}\rfloor.\end{aligned}$$

Proof. This follows from Theorem 1 by replacing every edge e of G by $w(e)$ parallel edges. □

Corollary 1 *The set of linear inequalities $x \ge 0$, $Mx \ge 1$ has the integer rounding property. That is, for every $w : E \to \mathbb{Z}_+$*

$$\begin{aligned}&\max\{y^T\mathbf{1} \mid y \ge 0, y^T M \le w, y \textit{ integral}\}\\ &\quad= \lfloor \max\{y^T\mathbf{1} \mid y \ge 0, y^T M \le w\}\rfloor\end{aligned}$$

Proof. Directly from Theorem 2 with (1). □

Corollary 1 is equivalent to: the polyhedron P has the integer decomposition property; that is, for each k, any integer vector in $k \cdot P$ is the sum of k integer vectors in P.

References

[1] F. Barahona, Packing spanning trees, *Math. Oper. Res.* **20** (1995), 104–115.

[2] R.P. Gupta, A decomposition theorem for bipartite graphs, in: *Theory of Graphs* (P. Rosenstiehl, ed.), Gordon and Breach, New York, 1967, 135–138

[3] J. Keijsper, An algorithm for packing connectors, report 97-9, Faculteit WINS, Universiteit van Amsterdam, 1997.

[4] J. Keijsper and R. Pendavingh, An efficient algorithm for minimum-weight bibranching, *J. Combin. Theory Ser. B*, to appear.

[5] D. Kőnig, Űber Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, *Math. Ann.* **77** (1916), 453–465.

[6] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, *J. London Math. Soc.* **36** (1961), 445–450.

[7] A. Schrijver, Min-max relations for directed graphs, *Ann. Discrete Math.* **16** (1982), 261–280.

[8] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, Chichester, 1986.

[9] W.T. Tutte, On the problem of decomposing a graph into n connected factors, *J. Londen Math. Soc.* **36** (1961), 221–230.

[10] D.J.A. Welsh, *Matroid theory* , Acad. Press, London, 1976.