Competition on the configuration model with infinite variance degrees

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Complex networks are large data-sets without obvious structure
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- elements are represented by *vertices*
Networks

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Complex networks are large data-sets without obvious structure

- elements are represented by *vertices*
- their relationship/interaction are represented by *edges*
- additional information can be added to vertices and edges
IP level internet network, 2003
from the OPTE project, opte.org
Some examples

- **Marketing:**
  - companies compete for customers
  - word-of-mouth recommendations on the acquaintance network
  - ‘online’ word-of-mouth: tweets, Facebook posts, etc.
Information spread and competition on networks

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- Epidemiology:
  - bacteria and viruses spread among population
  - different strains of a pathogen compete
Information spread and competition on networks

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- **Marketing:**
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**Coexistence?**

Can competitive spreading processes coexist on the network?
Can they both get linear proportion of the vertices?
Building a network: the configuration model
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\[ v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \]
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Degree assumptions

Degrees are i.i.d. copies of $D$, $P(D \geq 2) = 1$ and
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### Power law assumption

For $\tau \in (2, 3)$,

$$\frac{c_1}{x^{\tau-1}} \leq \mathbb{P}(D > x) \leq \frac{C_1}{x^{\tau-1}}$$
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**Power law assumption**

For $\tau \in (2, 3)$,

$$\frac{c_1}{x^{\tau-1}} \leq P(D > x) \leq \frac{C_1}{x^{\tau-1}}$$

This means, $E[D] < \infty$, but $E[D^2] = \infty$!
Competition starts!
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\[ t = 0 \]

\[ v_1 \]
\[ v_2 \]
\[ v_3 \]
\[ v_4 \]
\[ v_5 \]
\[ v_6 \]
\[ v_7 \]
\[ v_8 \]
Competition starts!

\[ t = 1 \]

\[
\begin{align*}
v_1 & \quad \quad v_2 \\
v_3 & \quad \quad v_4 \\
v_5 & \quad \quad v_6 \\
v_7 & \\
v_8 &
\end{align*}
\]
Competition starts!

\[ t = 2 \]

\[ v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_5 \rightarrow v_7 \]

\[ v_8 \rightarrow v_1 \]

\[ v_6 \rightarrow v_5 \rightarrow v_7 \]

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Competition starts!

\[ t = 3 \]

Graph with vertices labeled as follows:
- \( v_1 \)
- \( v_2 \)
- \( v_3 \)
- \( v_4 \)
- \( v_5 \)
- \( v_6 \)
- \( v_7 \)
- \( v_8 \)
Competition starts!

\[ t = 4 - \varepsilon \]
Competition starts!

\[ t = 4 \]

vertices

\[ v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \]
Competition starts!

\[ t = 4 \]

\[ v_1, v_2, v_3, v_4, v_5, v_8, v_7, v_6, v_3 \]
Different speeds: no coexistence

- Red gets $n - o(n)$ many vertices.
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Theorem (Baroni, v/d Hofstad, K), heuristic statement

$$\text{number of Blue nodes} \approx \exp \left\{ (\log n) \frac{2}{\lambda + 1} \cdot \xi_n \right\}$$ (1)
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- $\xi_n$ an oscillating, positive function of $n$,
- depends also on how ‘good’ the starting positions are.
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\text{number of Blue nodes } \approx \exp \left\{ (\log n) \frac{2}{\lambda + 1} \cdot \xi_n \right\} \tag{1}
\]

- $\xi_n$ an oscillating, positive function of $n$,
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No coexistence when the speeds differ.
Equal speeds: no coexistence with dissimilar neighbourhoods

If one of the starting neighbourhoods is ‘much better’ than the other (say red), then
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If one of the starting neighbourhoods is ‘much better’ than the other (say red), then

- red paints $n - o(n)$ vertices, and

Theorem (v/d Hofstad, K), heuristic statement

\[
\text{number of Blue vertices} \approx n^{\xi_n} \tag{2}
\]
Equal speeds: no coexistence with dissimilar neighbourhoods

If one of the starting neighbourhoods is ‘much better’ than the other (say red), then

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Theorem (v/d Hofstad, K), heuristic statement

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\text{number of Blue vertices} \approx n^{\xi_n} \quad (2)
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- $\xi_n < 1$ is an oscillating, positive function of $n$,
Equal speeds: no coexistence with dissimilar neighbourhoods

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Theorem (v/d Hofstad, K), heuristic statement

$$\text{number of Blue vertices} \approx n^{\xi_n} \quad (2)$$

- $\xi_n < 1$ is an oscillating, positive function of $n$,
- depends on how ‘different’ the starting positions are.
Equal speeds: coexistence with similar neighbourhoods

\( Y_r^{(n)}, Y_b^{(n)} \) are random variables that quantitatively describe how good the starting neighbourhoods are.
Equal speeds: coexistence with similar neighbourhoods

$Y_r^{(n)}, Y_b^{(n)}$ are random variables that quantitatively describe how good the starting neighbourhoods are.

**Theorem (v/d Hofstad, K)**

If $q := Y_r^{(n)}/Y_b^{(n)} \in (\tau - 2, (\tau - 2)^{-1})$, then there is coexistence.
Thank you for the attention!
Thank you for the attention!
Thank you for the attention!

\[ t = k^* + 1 \]

\[ n^{1/(\tau-1)} \]

\[ u_0 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \]

\[ n^{\tau-2} \quad n^{\tau-1} \]

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Thank you for the attention!

\[ t = k^* + 2 \]

\[ n^{1/(\tau-1)} \]
Thank you for the attention!
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$t = k^* + 4$

$n^{1/(\tau-1)}$

$u_0$

$u_1$

$u_2$

$u_3$

$u_4$

$u_5$

$\frac{\tau-2}{\tau-1}$

$\frac{\tau-1}{\tau-1}$

$u_0$

$u_1$

$u_2$

$u_3$

$u_4$

$u_5$
Thank you for the attention!
Thank you for the attention!

\[ t = T_r + 1 \]

\[ n^{1/(\tau-1)} \]
Thank you for the attention!

\[ t = T_r + 2 \]

\[ n^{1/(\tau - 1)} \]

\[ u_0, u_1, u_2, u_3, u_4, u_5 \]

\[ n^{\tau - 2}/n^{\tau - 1} \]
Thank you for the attention!
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Thank you for the attention!

\[ t = T_r + 5 \]

\[ n^{1/(\tau - 1)} \]

\[ u_0, u_1, u_2, u_3, u_4, u_5 \]

\[ n^{\frac{\tau - 2}{\tau - 1}} \]

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