Recap

- Morse Function
- Transversality
- Stable and Unstable Manifolds
- Morse-Smale Functions
A **Morse function** is a smooth function on a manifold, \( h : \mathbb{M} \rightarrow \mathbb{R} \), such that (i) all critical points are non-degenerate, and (ii) the critical points have distinct function values.

(ii) is sometimes dropped as it often is not required.
An **Integral Line** $\gamma$ on a manifold $\mathbb{M}$ is a maximal path $p$ whose tangent vectors agree with the gradient of the manifold.

\[
\text{org}(p) = \lim_{s \to -\infty} p(s)
\]
\[
\text{dest}(p) = \lim_{s \to \infty} p(s)
\]

Integral Lines have the following properties:

i. Any two integral lines are either disjoint or the same:

ii. Integral lines cover all of $\mathbb{M}$

iii. The limits $\text{org}(p)$ and $\text{dest}(p)$ are critical points of $h$
**Definition**

A **stable manifold** of a critical point $u$ of $h$ is the point itself together with all regular points whose integral lines end at $u$.

$$S(u) = \{u\} \cup \{x \in \mathbb{M} \mid \text{dest}(\gamma_x) = u\}$$

**Definition**

A **unstable manifold** of a critical point $u$ of $h$ is the point itself together with all regular points whose integral lines originate at $u$.

$$U(u) = \{u\} \cup \{y \in \mathbb{M} \mid \text{org}(\gamma_y) = u\}$$

Where $\gamma_x$ is the **integral line** that passes through point $x$. 
A **Morse-Smale function** is a Morse function, $h : M \to \mathbb{R}$, whose stable and unstable manifolds intersect transversally.
Morse-Smale Complexes
A Morse-Smale Complex is a collection of **Morse cells**.

Intersect all stable and unstable manifolds to obtain Morse cells.

Morse Cells are the connected components in the sets $U(a) \cap S(b)$, for all critical points $a, b \in \mathbb{M}$.

A Morse-Smale Complex subdivides a space into subspaces of similar 'flow'
Morse-Smale Complexes

Morse Cells are the connected components in the sets $U(a) \cap S(b)$, for all critical points $a, b \in \mathbb{M}$.
Observations (in 2-manifolds):

- The complex forms quadrilaterals with vertices of 0, 1, 2, 1 in that order.
- All minima and maxima are connected to two saddles points.
- All saddle points are connected to two minima and two maxima.
- The boundary of a stable manifold is a union of stable manifolds of lower dimensions.
Construct the complex iteratively:

- **0-skeleton**: Add all stable 0-manifolds (all minima).
- **1-skeleton**: Add all stable 1-manifolds. Each stable 1-manifold has a boundary of two cells from our 0-skeleton.
- **2-skeleton**: Add all stable 2-manifolds. The boundary of each of these manifolds is a set of cells from our 1-skeleton and our 0-skeleton.

The union of all skeletons forms our Morse-Smale complex.
We want a way of classifying the persistence (or importance) of a critical point:

- Pair up critical points such that they cancel each other out.
- Define a point’s persistence as the function distance between it and its paired point.
We want to pair up critical points:

- The Morse-Smale complex with edges between the minima and maxima forms a triangulation of the landscape.
- Build a complex by adding each critical point and its star to the complex in order of function value.
- Changes in connected components: minima create new components, maxima kill components. Saddles can be positive (create a component) or negative (kill a component).
We want to pair up critical points:

- Component is represented by 'lowest' point.
- When a negative point is added, it is paired with the 'highest' positive point from the two connected components.
- In the end 1 minimum, 1 maximum, and $\beta_1$ saddles remain unpaired.
We want a way of classifying the persistence (or importance) of a critical point:

- Pair up critical points such that they cancel each other out.
- Define a point's persistence as the function distance between it and its paired point.

**Definition**

The persistence of a critical point $a$ is the absolute height difference $p(a) = |h(b) - h(a)|$, if $a$ is paired with $b$, and $p(a) = \infty$ if $a$ is unpaired.
Hierarchy and Simplification

We can simplify our complex by eliminating two paired points together:

- Note a concept of hierarchy in the pairs of critical points (pairs are either nested or disjoint).
- Iteratively remove the lowest persistence pair (the least important critical points).
- Lowest persistence pair is always an innermost (lowest hierarchy) pair, we can keep repeating this until a desired level of simplification is reached.
Works similarly in 2-manifolds:
Morse-Smale function is a smooth function where all critical points are non-degenerate and stable and unstable manifolds intersect transversally.

- Iteratively build a Morse-Smale complex from stable manifolds from a Morse-Smale function.
- Use persistence based filtration to remove noise and/or simplify the resulting complex.
Morse-Smale Complexes for Piecewise Linear Manifolds [EHZ01][EHZ03]
Topology Toolkit [TTK]

