EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10. Date: Wednesday February 1, 2017. Time: 13:30–16:30. Place: AUD 9.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- The use of course notes is allowed, provided it is in immaculate state. The use of other notes, calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- Motivate your answers. You may provide your answers in Dutch or English.

GOOD LUCK!

(**30**) **1.** LINEAR ALGEBRA

Let V be an n-dimensional real vector space, with basis $\mathscr{B}_V = \{e_1, \ldots, e_n\}$.

($2\frac{1}{2}$) **a1.** Show that $o \notin \mathscr{B}_V$, i.e. the zero vector $o \in V$ is not admissible as a basis vector.

Suppose $o \in \mathscr{B}_V$, say $e_1 = o$. Then there exists a nontrivial linear combination $o = \lambda_1 e_1 + \ldots + \lambda_n e_n$, viz. take $\lambda_1 \neq 0$ arbitrary and $\lambda_2 = \ldots = \lambda_n = 0$. This contradicts the basic axiom of a basis.

($2\frac{1}{2}$) **a2.** Show that, for any given $v \in V$, its decomposition $v = v^1 e_1 + \ldots + v^n e_n$ relative to $\mathscr{B}_V$ is unique.

Suppose $v = v^1 e_1 + \ldots + v^n e_n = w^1 e_1 + \ldots + w^n e_n$, then $(v^1 - w^1)e_1 + \ldots + (v^n - w^n)e_n = o$. By definition of a basis $\mathscr{B}_V = \{e_1, \ldots, e_n\}$ this means that $v^i = w^i$ for each $i = 1, \ldots, n$.

Associated with V is the real *dual vector space* $V^* = \mathscr{L}(V, \mathbb{R})$ of all linear maps of type $\phi : V \to \mathbb{R}$.

(5) **b.** How is vector addition and scalar multiplication defined on V^*?

For any $\phi, \psi \in V^*$ and $\lambda, \mu \in \mathbb{R}$ we have $(\lambda\phi + \mu\psi)(v) = \lambda\phi(v) + \mu\psi(v)$ for all $v \in V$, which defines the vector structure on V^*.

We assume that V is equipped with a real inner product $\langle\, . \,|\, . \,\rangle : V \times V \to \mathbb{R} : (v, w) \mapsto \langle v|w\rangle$. Basis $\mathscr{B}_V$ and inner product $\langle\, . \,|\, . \,\rangle$ determine the so-called (non-singular) *Gram matrix* G, with entries

$$g_{ij} \doteq \langle e_i | e_j \rangle \,.$$

(5) **c.** Let $a = a^1 e_1 + \ldots + a^n e_n$ and $v = v^1 e_1 + \ldots + v^n e_n$. Show that $\langle a|v\rangle = \sum_{k,\ell=1}^n g_{k\ell} a^k v^\ell$.

We have $\langle a|v\rangle = \langle \sum_{k=1}^n a^k e_k | \sum_{\ell=1}^n v^\ell e_\ell \rangle \stackrel{*}{=} \sum_{k=1}^n \sum_{\ell=1}^n a^k v^\ell \langle e_k | e_\ell \rangle \stackrel{\star}{=} \sum_{k=1}^n \sum_{\ell=1}^n g_{k\ell} a^k v^\ell$. In $*$ we have used bilinearity of the real inner product, in $\star$ the definition of the entries of the Gram matrix.

(5) **d.** Show that the map $\phi_a : V \to \mathbb{R} : v \mapsto \phi_a(v) \doteq \langle a|v\rangle$ is linear, i.e. $\phi_a \in V^*$, for any $a \in V$.

For any $v, w \in V$ and $\lambda, \mu \in \mathbb{R}$ we have $\phi_a(\lambda v + \mu w) = \langle a|\lambda v + \mu w\rangle \stackrel{*}{=} \lambda\langle a|v\rangle + \mu\langle a|w\rangle = \lambda\phi_a(v) + \mu\phi_a(w)$. In $*$ we have used linearity of the inner product with respect to its second argument, the other equalities rely on the definition of the mapping $\phi_a \in V^*$.

(5) **e.** Show that $\phi_a \in V^*$ is uniquely defined by the parameter vector $a \in V$.
(*Hint:* Assume $\phi_a = \phi_b$ for $a, b \in V$, show that $a = b$.)

Suppose $\phi_a = \phi_b$ for some $a, b \in V$, i.e. $\phi_a(v) = \phi_b(v)$ for all $v \in V$, or $\langle a|v\rangle = \langle b|v\rangle$. Using (sesqui)linearity of the inner product this is equivalent to $\langle a - b|v\rangle = 0$. Since this holds for all $v \in V$, nondegeneracy of the inner product (consider e.g. $v = a - b$) implies $a - b = o$, i.e. $a = b$.

(5) **f.** Show that any $\phi \in V^*$ can be represented in the parametrised form $\phi_a = \langle a|\,.\,\rangle$ for some $a \in V$.
(*Hint:* Show that there exist coefficients $\phi_k \in \mathbb{R}$ such that $\phi(v) = \sum_{k=1}^n \phi_k v^k$, and find $a \in V$ in terms of these.)

Suppose $v = \sum_{k=1}^n v^k e_k \in V$ and $\phi \in V^*$, then $\phi(v) = \phi(\sum_{k=1}^n v^k e_k) \stackrel{*}{=} \sum_{k=1}^n v^k \phi(e_k) \stackrel{\text{def}}{=} \sum_{k=1}^n \phi_k v^k$. In $*$ we have made use of the linearity of $\phi \in V^*$, whereas the last identity defines the coefficients $\phi_k \stackrel{\text{def}}{=} \phi(e_k)$. In view of problem c let us take $a = \sum_{k=1}^n a^k e_k \in V$ such that $\langle a|v\rangle \stackrel{\text{c}}{=} \sum_{k=1}^n \sum_{\ell=1}^n g_{k\ell} a^k v^\ell = \sum_\ell \phi_\ell v^\ell$ for any $v \in V$. This suggets that we define $\phi_\ell \stackrel{\text{def}}{=} \sum_{k=1}^n g_{k\ell} a^k$. This is indeed a good definition, since the (symmetric) Gram matrix is invertible, so that we find an unambiguous vector $a \in V$ with coefficients $a^k = \sum_{\ell=1}^n g^{k\ell} \phi_\ell$, in which $g^{k\ell}$ denote the coefficients of the inverse Gram matrix.

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(**30**) **2.** GROUP THEORY (EXAM JANUARY 25, 2013, PROBLEM 1)

In this problem we consider the set of 2-parameter transformations on $\mathbb{L}_2(\mathbb{R})$ defined by

$$G = \{T_{a,b} : \mathbb{L}_2(\mathbb{R}) \to \mathbb{L}_2(\mathbb{R}) : f \mapsto T_{a,b}(f) \mid T_{a,b}(f)(x) = bf(x+a), a \in \mathbb{R}, b \in \mathbb{R}^+\} \,.$$

By $T_{a,b}(f)(x)$ we mean $(T_{a,b}(f))\,(x)$. We furnish the set G with the usual composition operator, indicated by the infix symbol $\circ$:

$$\circ : G \times G \to G : (T_{a,b}, T_{c,d}) \mapsto T_{a,b} \circ T_{c,d}\,,$$

i.e. $(T_{a,b} \circ T_{c,d})(f) = T_{a,b}(T_{c,d}(f))$.

a. Show that this is a good definition by proving the following claims for $a, c \in \mathbb{R}$, $b, d \in \mathbb{R}^+$:

(5) **a1.** If $f \in \mathbb{L}_2(\mathbb{R})$, then $T_{a,b}(f) \in \mathbb{L}_2(\mathbb{R})$. (Closure of $\mathbb{L}_2(\mathbb{R})$ under the mapping $T_{a,b}$.)

Suppose $f \in \mathbb{L}_2(\mathbb{R})$, then $\|T_{a,b}f\|_2^2 = \int_{-\infty}^{\infty} |bf(x+a)|^2 dx \overset{*}{=} b^2 \int_{-\infty}^{\infty} |f(y)|^2 dy = b^2\|f\|_2^2 < \infty$. In $*$ we have changed variables: $y = x + a$, the other identities rely on definitions.

(5) **a2.** If $T_{a,b} \in G$, then $T_{a,b} \circ T_{c,d} = T_{a+c,bd}$. (Closure of G under composition $\circ$.)

For arbitrary $f \in \mathbb{L}_2(\mathbb{R})$ and $x \in \mathbb{R}$ we have $(T_{a,b} \circ T_{c,d})(f)(x) = T_{a,b}(T_{c,d}(f))(x) = bT_{c,d}(f)(x+a) = bdf(x+a+c) = T_{a+c,bd}(f)(x)$, so that we may conclude that $T_{a,b} \circ T_{c,d} = T_{a+c,bd}$.

(10) **b.** Show that $\{G, \circ\}$ constitutes a commutative group, and give explicit expressions for the identity element $e \in G$, and for the inverse element $T_{a,b}^{\text{inv}} \in G$ corresponding to $T_{a,b} \in G$.

Closure has been proven. Let $a, c, e \in \mathbb{R}$ and $b, d, f \in \mathbb{R}^+$ be arbitrary. Commutativity follows from $T_{a,b} \circ T_{c,d} = T_{a+c,bd} = T_{c+a,db} = T_{c,d} \circ T_{a,b}$. Associativity follows from the fact that composition is, by construction, associative. Alternatively, exploit the identity proven under a2: $(T_{a,b} \circ T_{c,d}) \circ T_{e,f} = T_{a+c,bd} \circ T_{e,f} = T_{(a+c)+e,(bd)f} = T_{a+(c+e),b(df)} = T_{a,b} \circ (T_{c+e,df}) = T_{a,b} \circ (T_{c,d} \circ T_{e,f})$. The identity element is $T_{0,1} \in G$, since $T_{a,b} \circ T_{0,1} = T_{0,1} \circ T_{a,b} = T_{0+a,1b} = T_{a,b}$. The inverse element $T_{a,b}^{-1}$ of $T_{a,b} \in G$ is given by $T_{a,b}^{-1} = T_{-a,1/b} \in G$, since $T_{a,b} \circ T_{-a,1/b} = T_{-a,1/b} \circ T_{a,b} = T_{-a+a,b/b} = T_{0,1}$.

(10) **c.** Show that $G_1 = \{T_{a,b} \in G \mid a \in \mathbb{R}, b = 1\}$ is a subgroup of G.
(*Hint:* Exploit the fact that G is a group and $G_1 \subset G$.)

$T_{a,1} \circ T_{c,1} = T_{a+c,1} \in G_1$, so $G_1 \subset G$ is closed under $\circ$. Moreover, $T_{a,1}^{-1} = T_{-a,1} \in G$, so $G_1 \subset G$ is also closed under inversion. Hence $G_1 \subset G$ is a subgroup.

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(**20**) **3.** PDE THEORY AND FOURIER ANALYSIS

The so-called Bloch-Torrey equations describe the evolution of the 3 components of the magnetization vector field $\vec{M}(x,y,z,t) = (M_x(x,y,z,t), M_y(x,y,z,t), M_z(x,y,z,t))$ induced in a patient placed in an MRI scanner with static magnetic field $\vec{B}_0 = (0,0,B_0)$. In particular, the $\mathbb{C}$-valued transversal magnetization $m(x,y,z,t) \doteq M_x(x,y,z,t) + iM_y(x,y,z,t)$ satisfies the partial differential equation

$$\frac{\partial m}{\partial t} = -i\omega_0 m - \frac{m}{T_2} + D\Delta m\,,$$

in which $\Delta \doteq \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian. The so-called Larmor frequency $\omega_0 > 0$ is a constant proportional to B_0. We likewise assume $D > 0$, the diffusion coefficient, and $T_2 > 0$, the spin-spin relaxation time, to be constant.

(5) **a.** Give the corresponding evolution equation for $\widehat{m}(\omega_x, \omega_y, \omega_z, t)$ in the spatial Fourier domain.

We have

$$\frac{\partial \widehat{m}}{\partial t} = -\left(i\omega_0 + \frac{1}{T_2} + D\|\omega\|^2\right)\widehat{m}\,,$$

in which $\|\omega\|^2 \doteq \omega_x^2 + \omega_y^2 + \omega_z^2$.

At the start of the scan sequence, the system is initialized so that $m(x, y, z, t=0) = m_0(x, y, z)$, with Fourier transform $\widehat{m}_0(\omega_x, \omega_y, \omega_z)$.

(5) **b.** Determine $\widehat{m}(\omega_x, \omega_y, \omega_z, t)$ as a function of time $t \geq 0$, given $\widehat{m}_0(\omega_x, \omega_y, \omega_z)$.

We have

$$\widehat{m}(\omega_x, \omega_y, \omega_z, t) = \widehat{m}_0(\omega_x, \omega_y, \omega_z)\, e^{-(i\omega_0 + 1/T_2 + D\|\omega\|^2)t}\,.$$

c. Show that $\mu(t) \doteq \int_{\mathbb{R}^3} m(x, y, z, t)dxdydz$ is not preserved as a function of time by proving the following statements:

($2\frac{1}{2}$) **c1.** $|\mu(t)|$ decays exponentially over time towards zero.

($2\frac{1}{2}$) **c2.** $\mu(t)/|\mu(t)|$ rotates clockwise with uniform angular velocity around the origin of the $\mathbb{C}$-plane.

We have

$$\mu(t) = \int_{\mathbb{R}^3} m(x, y, z, t)dxdydz = \widehat{m}(0, 0, 0, t) = \widehat{m}_0(0, 0, 0)\, e^{-(i\omega_0 + 1/T_2)t}\,.$$

Both statements follow by inspection: $|\mu(t)| = |\widehat{m}_0(0, 0, 0)|\, e^{-t/T_2}$, resp. $\mu(t)/|\mu(t)| = e^{-i(\omega_0 t - \phi)}$, in which the phase angle ϕ is defined such that $e^{i\phi} \stackrel{\text{def}}{=} \widehat{m}_0(0, 0, 0)/|\widehat{m}_0(0, 0, 0)|$.

(5) **d.** Determine $m(x, y, z, t)$ as a function of time $t \geq 0$, given $m_0(x, y, z)$.

We have the solution implicitly in Fourier space, cf. problem b. Write it as

$$\widehat{m}(\omega_x, \omega_y, \omega_z, t) = e^{-(i\omega_0 + 1/T_2)t}\, \widehat{m}_0(\omega_x, \omega_y, \omega_z)\, \widehat{\phi}_t(\omega_x, \omega_y, \omega_z)\,,$$

in which

$$\widehat{\phi}_t(\omega_x, \omega_y, \omega_z) = e^{-D\|\omega\|^2 t}\,.$$

In order to obtain its Fourier inverse we may apply one of the convolution theorems (the overall factor $e^{-(i\omega_0 + 1/T_2)t}$ can be seen as a constant factor, since it does not involve $\omega = (\omega_x, \omega_y, \omega_z)$, and can be separated by virtue of linearity of the Fourier transform):

$$m(x, y, z, t) = e^{-(i\omega_0 + 1/T_2)t}\, \mathscr{F}^{-1}\left(\widehat{m}_0\, \widehat{\phi}_t\right)(x, y, z) = e^{-(i\omega_0 + 1/T_2)t}\,(m_0 * \phi_t)(x, y, z)\,,$$

in which

$$\phi_t(x, y, z) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-D\|\omega\|^2 t + i\omega\cdot x} d\omega = \frac{1}{\sqrt{4\pi Dt}^3}\, e^{-\frac{x^2 + y^2 + z^2}{4DT}}\,.$$

Here we have abbreviated $\omega \cdot x = \omega_x x + \omega_y y + \omega_z z$.

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(**20**) **4.** DISTRIBUTION THEORY & FOURIER ANALYSIS

The set notation $S_1 \subsetneq S_2$ means that S_1 is a strict subset of S_2, i.e. $S_1 \subset S_2$ and $S_1 \neq S_2$. Let $\mathscr{D}(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ be the class of Schwartz functions in one dimension that vanish outside a finite interval.

(5) **a.** Show that $\mathscr{D}(\mathbb{R}) \subsetneq \mathscr{S}(\mathbb{R})$ by providing an explicit example of an element $\phi \in \mathscr{S}(\mathbb{R})$ with $\phi \notin \mathscr{D}(\mathbb{R})$.

Take e.g. $\phi(x) = e^{-x^2}$. We have $\phi \in \mathscr{S}(\mathbb{R})$, but, since its essential domain is all of $\mathbb{R}$, $\phi \notin \mathscr{D}(\mathbb{R})$.

(5) **b.** Argue why this implies the strict inclusion $\mathscr{S}'(\mathbb{R}) \subsetneq \mathscr{D}'(\mathbb{R})$ for the corresponding topological duals.

If we restrict ourselves to regular distributions represented by 'functions under the integral', we may consider a function of non-polynomial growth, e.g. $f(x) = e^{x^2}$ (which cannot be bounded by a polynomial in the asymptotic regime $|x| \to \infty$). This function does define a distribution $F \in \mathscr{D}'(\mathbb{R})$, because the integral $F(\phi) = \int_{\mathbb{R}} f(x)\,\phi(x)\,dx$ does converge for all (compactly supported) $\phi \in \mathscr{D}(\mathbb{R})$, but it does not converge for all noncompactly supported $\phi \in \mathscr{S}(\mathbb{R})$ due to the asymptotic behaviour of $f(x)$ as $|x| \to \infty$. Consider e.g. the test function as in problem a, then the product function $f\phi$ defines a nonzero constant integrand that obviously cannot be integrated.

We take for granted that $\mathscr{S}(\mathbb{R})$ is closed under Fourier transformation as well as under convolution. Let $\mathscr{S}_m(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ be the subset of test functions $\phi \in \mathscr{S}(\mathbb{R})$ with $m+1$ vanishing momenta

$$\int_{-\infty}^{\infty} x^k\,\phi(x)\,dx = 0 \quad \text{for all } k=0,1,\ldots,m.$$

(5) **c.** Show that this integral condition for $\phi \in \mathscr{S}_m(\mathbb{R})$ is equivalent to $\widehat{\phi}^{(k)}(0)=0$ for all $k=0,1,\ldots,m$. (The parenthesised superscript indicates derivative order.)

By definition we have

$$\widehat{\phi}^{(k)}(\omega) = \frac{d^k}{d\omega^k}\widehat{\phi}(\omega) \stackrel{\text{def}}{=} \frac{d^k}{d\omega^k}\int_{-\infty}^{\infty} e^{-i\omega x}\,\phi(x)\,dx \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} (-ix)^k\,e^{-i\omega x}\,\phi(x)\,dx\,.$$

In particular this implies that

$$\widehat{\phi}^{(k)}(0) = \int_{-\infty}^{\infty} (-ix)^k\,\phi(x)\,dx\,,$$

which establishes the proof.

(5) **d.** Suppose $\phi \in \mathscr{S}_m(\mathbb{R})$ and $\psi \in \mathscr{S}_n(\mathbb{R})$ for some integers $m,n \geq 0$. Show that $\phi * \psi \in \mathscr{S}_p(\mathbb{R})$ in which $p=\max(m,n)$.

We have $\widehat{\phi * \psi} = \widehat{\phi}\,\widehat{\psi}$, so that, by virtue of the product rule,

$$\widehat{\phi * \psi}^{(k)}(\omega) = \left(\widehat{\phi}(\omega)\,\widehat{\psi}(\omega)\right)^{(k)} = \sum_{i=0}^{k}\binom{k}{i}\widehat{\phi}^{(i)}(\omega)\,\widehat{\psi}^{(k-i)}(\omega) \quad (k=0,1,2,\ldots)\,.$$

Inserting $\omega=0$ yields

$$\widehat{\phi * \psi}^{(k)}(0) = \sum_{i=0}^{k}\binom{k}{i}\widehat{\phi}^{(i)}(0)\,\widehat{\psi}^{(k-i)}(0)\,.$$

If $k \leq p = \max(m,n)$, then each term is trivial, since at least one of the factors vanishes, whence, as a result of a, $\phi * \psi \in \mathscr{S}_p(\mathbb{R})$.

THE END