EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10. Date: Wednesday February 1, 2017. Time: 13:30–16:30. Place: AUD 9.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- The use of course notes is allowed, provided it is in immaculate state. The use of other notes, calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- Motivate your answers. You may provide your answers in Dutch or English.

GOOD LUCK!

(**30**) **1.** LINEAR ALGEBRA

Let V be an n-dimensional real vector space, with basis $\mathscr{B}_V = \{e_1, \ldots, e_n\}$.

($2\frac{1}{2}$) **a1.** Show that $o \notin \mathscr{B}_V$, i.e. the zero vector $o \in V$ is not admissible as a basis vector.

($2\frac{1}{2}$) **a2.** Show that, for any given $v \in V$, its decomposition $v = v^1 e_1 + \ldots + v^n e_n$ relative to $\mathscr{B}_V$ is unique.

Associated with V is the real *dual vector space* $V^* = \mathscr{L}(V, \mathbb{R})$ of all linear maps of type $\phi : V \to \mathbb{R}$.

(5) **b.** How is vector addition and scalar multiplication defined on V^*?

We assume that V is equipped with a real inner product $\langle\, . \,|\, . \,\rangle : V \times V \to \mathbb{R} : (v, w) \mapsto \langle v|w\rangle$. Basis $\mathscr{B}_V$ and inner product $\langle\, . \,|\, . \,\rangle$ determine the so-called (non-singular) *Gram matrix* G, with entries

$$g_{ij} \doteq \langle e_i | e_j \rangle \, .$$

(5) **c.** Let $a = a^1 e_1 + \ldots + a^n e_n$ and $v = v^1 e_1 + \ldots + v^n e_n$. Show that $\langle a|v\rangle = \sum_{k,\ell=1}^{n} g_{k\ell} a^k v^\ell$.

(5) **d.** Show that the map $\phi_a : V \to \mathbb{R} : v \mapsto \phi_a(v) \doteq \langle a|v\rangle$ is linear, i.e. $\phi_a \in V^*$, for any $a \in V$.

(5) **e.** Show that $\phi_a \in V^*$ is uniquely defined by the parameter vector $a \in V$.
(*Hint:* Assume $\phi_a = \phi_b$ for $a, b \in V$, show that $a = b$.)

(5) **f.** Show that any $\phi \in V^*$ can be represented in the parametrised form $\phi_a = \langle a|\, . \,\rangle$ for some $a \in V$.
(*Hint:* Show that there exist coefficients $\phi_k \in \mathbb{R}$ such that $\phi(v) = \sum_{k=1}^{n} \phi_k v^k$, and find $a \in V$ in terms of these.)

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(**30**) **2.** GROUP THEORY (EXAM JANUARY 25, 2013, PROBLEM 1)

In this problem we consider the set of 2-parameter transformations on $\mathbb{L}_2(\mathbb{R})$ defined by

$$G = \{T_{a,b} : \mathbb{L}_2(\mathbb{R}) \to \mathbb{L}_2(\mathbb{R}) : f \mapsto T_{a,b}(f) \mid T_{a,b}(f)(x) = bf(x+a), a \in \mathbb{R}, b \in \mathbb{R}^+\} .$$

By $T_{a,b}(f)(x)$ we mean $(T_{a,b}(f))(x)$. We furnish the set G with the usual composition operator, indicated by the infix symbol $\circ$:

$$\circ : G \times G \to G : (T_{a,b}, T_{c,d}) \mapsto T_{a,b} \circ T_{c,d} ,$$

i.e. $(T_{a,b} \circ T_{c,d})(f) = T_{a,b}(T_{c,d}(f))$.

a. Show that this is a good definition by proving the following claims for $a, c \in \mathbb{R}$, $b, d \in \mathbb{R}^+$:

(5) **a1.** If $f \in \mathbb{L}_2(\mathbb{R})$, then $T_{a,b}(f) \in \mathbb{L}_2(\mathbb{R})$. (Closure of $\mathbb{L}_2(\mathbb{R})$ under the mapping $T_{a,b}$.)

(5) **a2.** If $T_{a,b} \in G$, then $T_{a,b} \circ T_{c,d} = T_{a+c,bd}$. (Closure of G under composition $\circ$.)

(10) **b.** Show that $\{G, \circ\}$ constitutes a commutative group, and give explicit expressions for the identity element $e \in G$, and for the inverse element $T^{\text{inv}}_{a,b} \in G$ corresponding to $T_{a,b} \in G$.

(10) **c.** Show that $G_1 = \{T_{a,b} \in G \mid a \in \mathbb{R}, b = 1\}$ is a subgroup of G.
(*Hint:* Exploit the fact that G is a group and $G_1 \subset G$.)

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(**20**) **3.** PDE THEORY AND FOURIER ANALYSIS

The so-called Bloch-Torrey equations describe the evolution of the 3 components of the magnetization vector field $\vec{M}(x,y,z,t) = (M_x(x,y,z,t), M_y(x,y,z,t), M_z(x,y,z,t))$ induced in a patient placed in an MRI scanner with static magnetic field $\vec{B}_0 = (0,0,B_0)$. In particular, the $\mathbb{C}$-valued transversal magnetization $m(x,y,z,t) \doteq M_x(x,y,z,t) + iM_y(x,y,z,t)$ satisfies the partial differential equation

$$\frac{\partial m}{\partial t} = -i\omega_0 m - \frac{m}{T_2} + D\Delta m ,$$

in which $\Delta \doteq \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian. The so-called Larmor frequency $\omega_0 > 0$ is a constant proportional to B_0. We likewise assume $D > 0$, the diffusion coefficient, and $T_2 > 0$, the spin-spin relaxation time, to be constant.

(5) **a.** Give the corresponding evolution equation for $\widehat{m}(\omega_x, \omega_y, \omega_z, t)$ in the spatial Fourier domain.

At the start of the scan sequence, the system is initialized so that $m(x,y,z,t=0) = m_0(x,y,z)$, with Fourier transform $\widehat{m}_0(\omega_x, \omega_y, \omega_z)$.

(5) **b.** Determine $\widehat{m}(\omega_x, \omega_y, \omega_z, t)$ as a function of time $t \geq 0$, given $\widehat{m}_0(\omega_x, \omega_y, \omega_z)$.

c. Show that $\mu(t) \doteq \int_{\mathbb{R}^3} m(x,y,z,t)dxdydz$ is not preserved as a function of time by proving the following statements:

($2\frac{1}{2}$) **c1.** $|\mu(t)|$ decays exponentially over time towards zero.

($2\frac{1}{2}$) **c2.** $\mu(t)/|\mu(t)|$ rotates clockwise with uniform angular velocity around the origin of the $\mathbb{C}$-plane.

(5) **d.** Determine $m(x, y, z, t)$ as a function of time $t \geq 0$, given $m_0(x, y, z)$.

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(**20**) **4.** DISTRIBUTION THEORY & FOURIER ANALYSIS

The set notation $S_1 \subsetneq S_2$ means that S_1 is a strict subset of S_2, i.e. $S_1 \subset S_2$ and $S_1 \neq S_2$. Let $\mathscr{D}(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ be the class of Schwartz functions in one dimension that vanish outside a finite interval.

(5) **a.** Show that $\mathscr{D}(\mathbb{R}) \subsetneq \mathscr{S}(\mathbb{R})$ by providing an explicit example of an element $\phi \in \mathscr{S}(\mathbb{R})$ with $\phi \notin \mathscr{D}(\mathbb{R})$.

(5) **b.** Argue why this implies the strict inclusion $\mathscr{S}'(\mathbb{R}) \subsetneq \mathscr{D}'(\mathbb{R})$ for the corresponding topological duals.

We take for granted that $\mathscr{S}(\mathbb{R})$ is closed under Fourier transformation as well as under convolution. Let $\mathscr{S}_m(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ be the subset of test functions $\phi \in \mathscr{S}(\mathbb{R})$ with $m+1$ vanishing momenta

$$\int_{-\infty}^{\infty} x^k \, \phi(x) \, dx = 0 \quad \text{for all } k = 0, 1, \ldots, m.$$

(5) **c.** Show that this integral condition for $\phi \in \mathscr{S}_m(\mathbb{R})$ is equivalent to $\widehat{\phi}^{(k)}(0) = 0$ for all $k = 0, 1, \ldots, m$. (The parenthesised superscript indicates derivative order.)

(5) **d.** Suppose $\phi \in \mathscr{S}_m(\mathbb{R})$ and $\psi \in \mathscr{S}_n(\mathbb{R})$ for some integers $m, n \geq 0$. Show that $\phi * \psi \in \mathscr{S}_p(\mathbb{R})$ in which $p = \max(m, n)$.

THE END