EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10. Date: Saturday April 16, 2020. Time: 13:30–16:30. Canvas online assignment.

READ THIS FIRST!

- Write your name and student identification number on each paper, and include it in the file name(s) of any scan you upload via Assignments in Canvas.
- The exam consists of 4 problems and a mandatory legal statement. Credits are indicated in the margin.
- For the online written exam, *address problems 1, 2 and 3 only*. *Ignore the old exam problem 4* (this problem may be a topic for your oral exam afterwards).
- Follow the legal statement instruction at the end of the exam before you scan and upload your results! *This requires you to include a written statement exactly as indicated in that clause.*
- You may consult the online course notes "Mathematical Techniques for Image Analysis (2DMM10)" on your laptop. No other material or equipment may be used.
- The host/invigilator may ask you to temporarily share your screen *promptly without undue delay* at any given moment via a chat request directly to you. Your screen should display no other items than those permitted for this exam (online course notes, Zoom, Canvas).
- Keep your camera on and do not unmute yourself, as this may be distractive.
- You can chat with the host/invigilator (only) for urgent requests. Do not step away from the camera without such a request and before explicit approval. Use this option only if strictly necessary. In principle, only one short sanitary break will be allowed during the exam.

GOOD LUCK!

(35) **1.** VECTOR SPACE

We consider the subset H of points in $\mathbb{R}^3$ given by

$$(\star) \qquad \mathrm{H} \doteq \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2+y^2=1 \right\} \,,$$

equipped with a binary operation $\oplus : \mathrm{H} \times \mathrm{H} \to \mathrm{H}$, viz.

$$(x,y,z) \oplus (u,v,w) \doteq (xu - yv, yu + xv, z + w)\,.$$

Note that an element $X = (x,y,z) \in \mathrm{H}$ may be reparametrized in terms of cylindrical coordinates:

$$(x,y,z) = (\cos\phi, \sin\phi, z)\,,$$

in which $\phi \in \mathbb{R}$, which has the advantage that the constraint $r^2 \doteq x^2+y^2=1$ in $(\star)$ is automatic. You may use the following lemmas without proof.

Lemma 1. For $\phi, \theta \in \mathbb{R}$ we have

$$\begin{aligned} \cos(\phi \pm \theta) &= \cos\phi \, \cos\theta \mp \sin\phi \, \sin\theta \,, \\ \sin(\phi \pm \theta) &= \sin\phi \, \cos\theta \pm \cos\phi \, \sin\theta \,. \end{aligned}$$

Lemma 2. The set H, furnished with the binary operator $\oplus$, constitutes a commutative group.

In an attempt to define a scalar multiplication we consider the operator $\odot : \mathbb{R} \times \mathrm{H} \to \mathrm{H}$, given by

$$(\dagger) \qquad \lambda \odot X \doteq (\cos(\lambda\phi), \sin(\lambda\phi), \lambda z) \,,$$

in which $X \doteq (\cos\phi, \sin\phi, z)$ as before, however with the restriction that $\phi \in [0, 2\pi)$.

Conjecture. The commutative group $\{\mathrm{H} \,, \oplus\}$, additionally furnished with the external operator $\odot$ according to the definition $(\dagger)$ above, constitutes a vector space.

a. Prove this conjecture, as follows:

(5) **a1.** Prove $\lambda \odot (X \oplus U) = (\lambda \odot X) \oplus (\lambda \odot U)$.

(5) **a2.** Prove $(\lambda + \mu) \odot X = (\lambda \odot X) \oplus (\mu \odot X)$.

(5) **a3.** Prove $(\lambda\mu) \odot X = \lambda \odot (\mu \odot X)$.

(5) **a4.** Prove $1 \odot X = X$.

By virtue of Lemma 1 we have for $X \doteq (\cos\phi, \sin\phi, z) \in \mathrm{H}$ and $U \doteq (\cos\theta, \sin\theta, w) \in \mathrm{H}$,

$$(\bullet) \qquad X \oplus U = (\cos(\phi+\theta), \sin(\phi+\theta), z + w) \,,$$

which will be used in all manipulations below. In this expression, by virtue of 2π-periodicity, the angular sum $\phi + \theta$ may be understood as '$\phi+\theta \pmod{2\pi}$', i.e. as the smallest number $\alpha \in [0, 2\pi)$ such that $\phi + \theta = \alpha + k \cdot 2\pi$ for some integer $k \in \mathbb{Z}$. This will be tacitly assumed for all arguments of sin and cos below. (The reason for this assumption lies in the restriction for the applicability of $(\dagger)$ in Lemma 2.)

a1. We have, using $(\bullet)$, $(\dagger)$ and $X \doteq (\cos\phi, \sin\phi, z)$, $U \doteq (\cos\theta, \sin\theta, w)$, $\lambda \in \mathbb{R}$,

$$\begin{aligned} \lambda \odot (X \oplus U) &\stackrel{\bullet}{=} \lambda \odot (\cos(\phi+\theta), \sin(\phi+\theta), z + w) \stackrel{\dagger}{=} (\cos(\lambda(\phi+\theta)), \sin(\lambda(\phi+\theta)), \lambda(z+w)) = \\ &(\cos(\lambda\phi + \lambda\theta), \sin(\lambda\phi + \lambda\theta), \lambda z + \lambda w) = (\cos(\lambda\phi), \sin(\lambda\phi), \lambda z) \oplus (\cos(\lambda\theta), \sin(\lambda\theta), \lambda w) = \\ &(\lambda \odot (\cos\phi, \sin\phi, z)) \oplus (\lambda \odot (\cos\theta, \sin\theta, w)) \doteq \\ &(\lambda \odot X) \oplus (\lambda \odot U) \,. \end{aligned}$$

a2. Furthermore, if $\mu \in \mathbb{R}$ is another scalar, then

$$\begin{aligned} (\lambda + \mu) \odot X &\stackrel{\dagger}{=} (\cos((\lambda+\mu)\phi), \sin((\lambda+\mu)\phi), (\lambda+\mu)z) = (\cos(\lambda\phi + \mu\phi), \sin(\lambda\phi + \mu\phi), \lambda z + \mu z) \stackrel{\bullet}{=} \\ &(\cos(\lambda\phi), \sin(\lambda\phi), \lambda z) \oplus (\cos(\mu\phi), \sin(\mu\phi), \mu z) \stackrel{\dagger}{=} (\lambda \odot (\cos\phi, \sin\phi, z)) \oplus (\mu \odot (\cos\phi, \sin\phi, z)) \doteq \\ &(\lambda \odot X) \oplus (\mu \odot X) \,. \end{aligned}$$

a3. Also,

$$\begin{aligned} (\lambda\mu) \odot X &\stackrel{\dagger}{=} (\cos((\lambda\mu)\phi), \sin((\lambda\mu)\phi), (\lambda\mu)z) = (\cos(\lambda(\mu\phi)), \sin(\lambda(\mu\phi)), \lambda(\mu z)) \stackrel{\dagger}{=} \\ &\lambda \odot (\cos(\mu\phi), \sin(\mu\phi), \mu z) \stackrel{\dagger}{=} \lambda \odot (\mu \odot (\cos\phi, \sin\phi, z)) \doteq \\ &\lambda \odot (\mu \odot X) \,. \end{aligned}$$

a4. Finally, we have

$$1 \odot X \doteq 1 \odot (\cos\phi, \sin\phi, z) \stackrel{\dagger}{=} (\cos(1\cdot\phi), \sin(1\cdot\phi), 1\cdot z) = (\cos\phi, \sin\phi, z) \doteq X\,.$$

We consider the unit pitch helix $\mathrm{H}_1 \subset \mathrm{H}$, defined as $\mathrm{H}_1 \doteq \{(\cos(2\pi t), \sin(2\pi t), t) \mid t \in \mathbb{R}\}$.

(10) **b.** Prove that H_1 is itself a vector space.
(*Hint:* Exploit Lemma 2 and the Conjecture above.)

The hint refers to the Subspace Theorem, which states that if $\mathrm{H}_1 \subset \mathrm{H}$ is closed under $\oplus$ and $\odot$, then it is a vector space. To prove this, take $X \doteq (\cos(2\pi\phi), \sin(2\pi\phi), \phi) \in \mathrm{H}_1$, $U \doteq (\cos(2\pi\theta), \sin(2\pi\theta), \theta) \in \mathrm{H}_1$, $\lambda, \mu \in \mathbb{R}$. Indeed, we have

$$(\lambda\odot X)\oplus(\mu\odot U) \stackrel{\dagger}{=} (\cos(2\pi\lambda\phi), \sin(2\pi\lambda\phi), \lambda\phi)\oplus(\cos(2\pi\mu\theta), \sin(2\pi\mu\theta), \mu\theta) \stackrel{\bullet}{=} (\cos(2\pi(\lambda\phi+\mu\theta)), \sin(2\pi(\lambda\phi+\mu\theta)), \lambda\phi+\mu\theta) \in \mathrm{H}_1\,.$$

(5) **c.** Consider the vector $V \in \mathrm{H}_1$ given by $V = (1,0,1)$. Show that $t \odot V = (1,0,t) \in \mathrm{H}_1$ if and only if $t \in \mathbb{Z}$.

Assume $k \in \mathbb{Z}$. We have $t\odot V \doteq t\odot(1,0,1) \stackrel{!}{=} t\odot(\cos(0), \sin(0), 1) \stackrel{\dagger}{=} (\cos(t\cdot 0), \sin(t\cdot 0), t\cdot 1) = (\cos(2\pi k), \sin(2\pi k), t) = (1,0,t)$. The ! indicates the caveat that when representing a vector in cylindrical coordinates one must restrict the polar angle to the half open interval $[0, 2\pi)$ in order to apply $(\dagger)$. The last equality shows that $t \in \mathbb{R}$ must be equal to some $k \in \mathbb{Z}$ in order for $(1, 0, t \doteq k) = (\cos(2\pi k), \sin(2\pi k), k) \in \mathrm{H}_1$.

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(20) 2. Inner Product

For $(x,y) \in \mathbb{R}^2$ and $(u,v) \in \mathbb{R}^2$ we stipulate an inner product of the form

$$(x,y)\cdot(u,v) \doteq xu + a(xv + yu) + yv\,,$$

in which $a \in \mathbb{R}$ is a parameter.

- Find all a for which this definition is consistent with the axioms of a real inner product. Prove your result.

We have to check bilinearity, commutativity, positivity and nondegeneracy. Linearity w.r.t. the first vector (x,y) follows by writing the inner product as a function of the components x and y for fixed parameters u, v and a, viz. $(x,y)\cdot(u,v) \doteq \ell_{a,u,v}(x,y) = (u+av)x + (v+au)y$. Clearly $\ell_{a,u,v}$ is a linear function of x and y for any choice of $a, u, v \in \mathbb{R}$. A similar argument applies to linearity w.r.t. the second vector, but this also follows from the above i.c.w. commutativity, v.i. Commutativity is obvious from the form invariance under interchange of $(x,y) \leftrightarrow (u,v)$ and holds also regardless of the choice of $a \in \mathbb{R}$. Positivity and nondegeneracy follow from inspection of the quadratic function $Q_a(\xi) \doteq \xi^2 + 2a\xi + 1$, which arises by setting $Q_a(\xi) = (x,y)\cdot(x,y)/y^2$ if $y \neq 0$, with $\xi \doteq x/y$, or by setting $Q_a(\xi) = (x,y)\cdot(x,y)/x^2$ if $x \neq 0$, with, in that case, $\xi \doteq y/x$. Note that if $(x,y) \neq (0,0)$, then we can use one of the two definitions above. The function Q_a (corresponding to an upward opening parabola) must have no zeros in order to be positive definite, whence its discriminant must be negative: $D = 4a^2 - 4a < 0$, i.e. $-1 < a < 1$. We then have $(x,y)\cdot(x,y) = 0$ iff $(x,y) = (0,0)$.

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(15) 3. Fourier Transformation and Convolution

(5) **a.** Show that $*$ is closed on $L^1(\mathbb{R}^n)$, i.e. show that $f * g \in L^1(\mathbb{R}^n)$ for all $f, g \in L^1(\mathbb{R}^n)$.

Young Inequality states $\|f * g\|_r \leq \|f\|_p \, \|g\|_q$ if the right hand side norms exist (including the cases $p, q = \infty$) and if $1 \leq r \leq \infty$ is such that $1/p + 1/q = 1 + 1/r$. In view of the assumptions take $p = q = r = 1$, which is indeed compatible with the theorem.

You may use the following lemma without proof.

Lemma. Fourier transformation is well-defined on $L^1(\mathbb{R}^n)$. Moreover, if $f \in L^1(\mathbb{R}^n)$, then the Fourier transform $\hat{f} \in C(\mathbb{R}^n)$ is (uniformly) continuous.

(5) **b.** Show that the Fourier transform of $f \in L^1(\mathbb{R}^n)$ is bounded, i.e. $\hat{f} \in L^\infty(\mathbb{R}^n)$.

We have

$$|\hat{f}(\omega)| \doteq \left| \int_{\mathbb{R}^n} e^{-i\omega \cdot x} f(x) \, dx \right| \leq \int_{\mathbb{R}^n} \left| e^{-i\omega \cdot x} f(x) \right| \, dx = \int_{\mathbb{R}^n} |f(x)| \, dx \doteq \|f\|_1 \, .$$

Thus also $\|\hat{f}\|_\infty \doteq \sup_{\omega \in \mathbb{R}^n} |\hat{f}(\omega)| \leq \|f\|_1$.

We furthermore assume that $f \geq 0$, $g \geq 0$, by which we mean that $f(x) \geq 0$, $g(x) \geq 0$ for all $x \in \mathbb{R}^n$ for which these functions are defined.

(5) **c.** Show that in this case we have strict equality $\|f * g\|_1 = \|f\|_1 \, \|g\|_1$.

Nonnegativity $f \geq 0$, $g \geq 0$ implies nonnegativity $f * g \geq 0$, and moreover $\|f\|_1 = \hat{f}(0)$, $\|g\|_1 = \hat{g}(0)$, whence

$$\|f * g\|_1 = \int_{\mathbb{R}^n} (f * g)(x) \, dx = (\widehat{f * g})(0) = \hat{f}(0) \, \hat{g}(0) = \|f\|_1 \, \|g\|_1 \, .$$

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(30) 4. Distribution Theory (Exam January 25, 2013, Problem 2)

Recall the Dirac point distribution,

$$\delta : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto \delta(\phi) = \phi(0) \, ,$$

and its derivative,

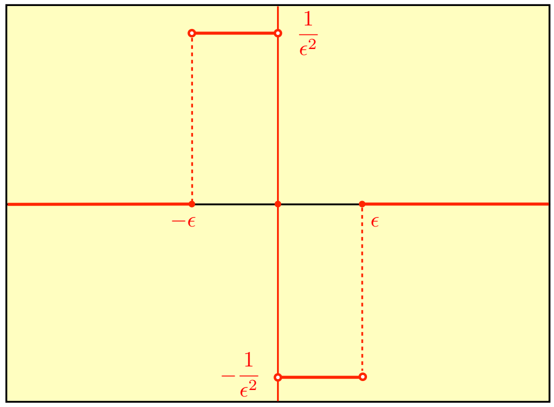
$$\delta' : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto \delta'(\phi) = -\phi'(0) \, .$$

In this problem we consider an approximation of δ' in the form of a 1-parameter family of functions, given by

$$f_\epsilon : \mathbb{R} \to \mathbb{R} : x \mapsto f_\epsilon(x) = \begin{cases} 0 & \text{if } x \leq -\epsilon \\ 1/\epsilon^2 & \text{if } -\epsilon < x < 0 \\ 0 & \text{if } x = 0 \\ -1/\epsilon^2 & \text{if } 0 < x < \epsilon \\ 0 & \text{if } x \geq \epsilon \end{cases}$$

with $\epsilon > 0$.

(5) **a.** Draw the graph of $y = f_\epsilon(x)$, indicating relevant values (in terms of ϵ) on each axis.



Consider the regular tempered distribution T_{f_ϵ} associated with the function f_ϵ, i.e.

$$T_{f_\epsilon} : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_{f_\epsilon}(\phi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)dx\,.$$

(10) **b.** Show that $T_{f_\epsilon}(\phi) = \dfrac{1}{\epsilon^2}\displaystyle\int_{-\epsilon}^{0} \phi(x)dx - \dfrac{1}{\epsilon^2}\int_0^\epsilon \phi(x)dx$.

We may replace the integration boundaries $\pm\infty$ by the boundaries $\pm\epsilon$ of the compact domain of support of f_ϵ. Subsequently we may split the integral $\int_{-\epsilon}^{\epsilon} = \int_{-\epsilon}^{0} + \int_0^\epsilon$, and substitute the definition of f_ϵ.

Let $\phi \in \mathscr{S}(\mathbb{R})$ be an analytical test function, and recall Taylor's theorem:

$$\phi(x) = \phi(0) + \phi'(0)x + \frac{1}{2}\phi''(\xi)x^2\,,$$

in which ξ is some number between 0 and x.

(5) **c.** Using this Taylor expansion, argue (mathematically) why we may replace the Lagrange remainder term $\frac{1}{2}\phi''(\xi)x^2$ in the expression for $T_{f_\epsilon}(\phi)$ by a term of order $\mathcal{O}(\epsilon^2)$ as $\epsilon \downarrow 0$.

Both ξ and x are of order $\mathcal{O}(\epsilon)$ on the effective support interval, so $\phi''(\xi) = \mathcal{O}(1)$, whence we have for the Lagrange remainder $\frac{1}{2}\phi''(\xi)x^2 = \mathcal{O}(\epsilon^2)$.

(10) **d.** Show that, for any analytical test function $\phi \in \mathscr{S}(\mathbb{R})$, $\lim\limits_{\epsilon\downarrow 0} T_{f_\epsilon}(\phi) = \delta'(\phi)$.

We have

$$\lim_{\epsilon\downarrow 0} T_{f_\epsilon}(\phi) = \lim_{\epsilon\downarrow 0}\left[\frac{1}{\epsilon^2}\int_{-\epsilon}^{0}\phi(x)dx - \frac{1}{\epsilon^2}\int_0^\epsilon \phi(x)dx\right] = \lim_{\epsilon\downarrow 0}\left[\frac{1}{\epsilon^2}\int_{-\epsilon}^{0}\left(\phi(0)+\phi'(0)x+\mathcal{O}(\epsilon^2)\right)dx - \frac{1}{\epsilon^2}\int_0^\epsilon\left(\phi(0)+\phi'(0)x+\mathcal{O}(\epsilon^2)\right)dx\right].$$

Observe the following:

- The zeroth order terms in both integrals cancel eachother.
- The first order terms in both integrals add up: $\frac{1}{\epsilon^2}\int_{-\epsilon}^{0}\phi'(0)x dx - \frac{1}{\epsilon^2}\int_{0}^{\epsilon}\phi'(0)x dx = (-\frac{1}{2}-\frac{1}{2})\phi'(0) = -\phi'(0)$.
- The $\mathcal{O}(\epsilon^2)$ term in each integral yields, after integration, and taking into account the factor $1/\epsilon^2$, an $\mathcal{O}(\epsilon)$ outcome.

Thus

$$\lim_{\epsilon\downarrow 0} T_{f_\epsilon}(\phi) = \lim_{\epsilon\downarrow 0}\left[-\phi'(0)+\mathcal{O}(\epsilon)\right] = -\phi'(0) \stackrel{\text{def}}{=} \delta'(\phi)\,.$$

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☞ **INSTRUCTION FOR LEGAL STATEMENT.** Please read the following paragraph carefully, and copy the text below it verbatim to your answer sheet.

By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

Text to be copied (with optional remarks):

✍ *I made this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.*

Remarks from the student: ....

THE END