EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10. Date: Friday January 24, 2020. Time: 09:00–12:00. Place: Vertigo 4.06 A.

READ THIS FIRST!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- The use of course notes is allowed, provided it is in immaculate state. The use of other notes, calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- Motivate your answers. You may provide your answers in Dutch or English.

GOOD LUCK!

(35) **1. VECTOR SPACE**

We consider the subset H of points in $\mathbb{R}^3$ given by

$$(\star) \qquad \mathrm{H} \doteq \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2+y^2=1\right\} ,$$

equipped with internal and external operations $\oplus : \mathrm{H}\times\mathrm{H} \to \mathrm{H}$, respectively $\otimes : \mathbb{R}\times\mathrm{H} \to \mathrm{H}$, viz.

$$\begin{aligned}(x,y,z)\oplus(u,v,w) &\doteq (xu-yv, yu+xv, z+w)\,,\\ \lambda\otimes(x,y,z) &\doteq (x\cos(2\pi\lambda)-y\sin(2\pi\lambda), y\cos(2\pi\lambda)+x\sin(2\pi\lambda), z+\lambda)\,.\end{aligned}$$

For notational convenience we abbreviate elements of H as $X \doteq (x,y,z)$, $U \doteq (u,v,w)$ et cetera.

(5) **a.** Prove closure, i.e. show that $X \oplus U \in \mathrm{H}$ and $\lambda \otimes X \in \mathrm{H}$ for all $X, U \in \mathrm{H}$ and $\lambda \in \mathbb{R}$.

Let $X=(x,y,z)\in\mathrm{H}$, $U=(u,v,w)\in\mathrm{H}$, $\lambda\in\mathbb{R}$. Then $X\oplus U \doteq (x,y,z)\oplus(u,v,w) \doteq (xu-yv, yu+xv, z+w)$ satisfies the constraint in $(\star)$ for its first two entries, since

$$(xu-yv)^2+(yu+xv)^2 = (x^2+y^2)(u^2+v^2) \doteq 1\,,$$

in which the last identity holds by definition of $X, U \in \mathrm{H}$. The closure requirement on the third entry is trivially fulfilled: $z+w\in\mathbb{R}$ if $z,w\in\mathbb{R}$. Also, $\lambda\otimes X$ satisfies the constraint in $(\star)$ for its first two entries, since

$$(x\cos(2\pi\lambda)-y\sin(2\pi\lambda))^2+(y\cos(2\pi\lambda)+x\sin(2\pi\lambda))^2 = (x^2+y^2)(\cos^2(2\pi\lambda)+\sin^2(2\pi\lambda)) = 1\,,$$

while the closure requirement on the third entry is again trivial: $z+\lambda\in\mathbb{R}$ if $z,\lambda\in\mathbb{R}$.

(5) **b.** Show that $X \in \mathrm{H}$ can be parametrized such that the constraint in $(\star)$ is automatically fulfilled.
(*Hint:* Introduce an angle $\phi\in\mathbb{R}$ and consider polar coordinates for the (x,y)-plane.)

Writing $X=(x,y,z)\doteq(\cos\phi,\sin\phi,z)$ will enforce the constraint $x^2+y^2=\cos^2\phi+\sin^2\phi=1$ automatically. Note that any $(x,y,z)\in\mathrm{H}$ can be represented in this way for some (unique) $z\in\mathbb{R}$ and some (non-unique) polar angle $\phi\in\mathbb{R}$.

We now investigate whether $(\star)$, furnished with the operators $\oplus$ and $\otimes$, satisfies all vector space axioms. We consider the abelian group requirement for $\oplus$ first.

You may use the following lemma.

Lemma. For $\phi, \theta \in \mathbb{R}$ we have

$$\cos(\phi \pm \theta) = \cos\phi \cos\theta \mp \sin\phi \sin\theta \,,$$
$$\sin(\phi \pm \theta) = \sin\phi \cos\theta \pm \cos\phi \sin\theta \,.$$

(5) **c1.** Prove associativity: $(X \oplus U) \oplus A = X \oplus (U \oplus A)$ for all $X, U, A \in \mathrm{H}$.
(*Hint:* Exploit your observation in b and use the lemma.)

Take polar angles such that $X = (x, y, z) \doteq (\cos\phi, \sin\phi, z)$, $U = (u, v, w) \doteq (\cos\theta, \sin\theta, w)$, $A = (a, b, c) \doteq (\cos\psi, \sin\psi, c)$. The crucial observation is that, by virtue of the lemma,

$$(\bullet) \qquad X \oplus U = (\cos(\phi{+}\theta), \sin(\phi{+}\theta), z + w)\,.$$

Tacitly using trivial properties, notably associativity, of $\{\mathbb{R}, +\}$ at various places, besides the consequence of the lemma in the form of $(\bullet)$ and the definition of $\oplus$, we thus obtain

$$\begin{aligned}(X \oplus U) \oplus A &= (\,(\cos(\phi{+}\theta), \sin(\phi{+}\theta), z + w)\,) \oplus (\cos\psi, \sin\psi, c) = \\ &(\cos((\phi + \theta) + \psi), \sin((\phi + \theta) + \psi), (z + w) + c) = (\cos(\phi + (\theta + \psi)), \sin(\phi + (\theta + \psi)), z + (w + c)) = \\ &(\cos\phi, \sin\phi, z) \oplus (\,(\cos(\theta + \psi), \sin(\theta + \psi), w + c)\,) = X \oplus (U \oplus A)\,.\end{aligned}$$

($2\frac{1}{2}$) **c2.** Prove commutativity: $X \oplus U = U \oplus X$ for all $X, U \in \mathrm{H}$.

This is a direct consequence of the lemma, notably $(\bullet)$, and trivial properties of $\{\mathbb{R}, +\}$, notably its commutativity:

$$X \oplus U = (\cos(\phi{+}\theta), \sin(\phi{+}\theta), z + w) = (\cos(\theta{+}\phi), \sin(\theta{+}\phi), w + z) = U \oplus X\,.$$

(A direct proof based on $(\star)$, i.e. not using the polar representation, is equally straightforward.)

($2\frac{1}{2}$) **c3.** Show that $E \doteq (1, 0, 0) \in \mathrm{H}$ is the neutral element for $\oplus$.

With the help of $(\bullet)$ and the observation that $E = (\cos 0, \sin 0, 0)$, a direct computation reveals that, for any $X = (\cos\phi, \sin\phi, z) \in \mathrm{H}$,

$$X \oplus E = (\cos\phi, \sin\phi, z) \oplus (\cos 0, \sin 0, 0) = (\cos(\phi + 0), \sin(\phi + 0), z + 0) = (\cos\phi, \sin\phi, z) = X\,.$$

By commutativity (c2) we then also have $E \oplus X = X \oplus E = X$. (A direct proof based on $(\star)$, i.e. not using the polar representation, is equally straightforward.)

(5) **c4.** State the explicit form of the antivector $(-X) \in \mathrm{H}$ for any given $X \in \mathrm{H}$, and prove $(-X) \oplus X = E$.

Inspired by the polar form, $X = (\cos\phi, \sin\phi, z)$, replace $\phi \in \mathbb{R}$ by $-\phi \in \mathbb{R}$ and $z \in \mathbb{R}$ by $-z \in \mathbb{R}$, i.e. stipulate $(-X) = (\cos(-\phi), \sin(-\phi), -z)$. Indeed, we then have, using $(\bullet)$ once again, as well the trivialities of $\{\mathbb{R}, +\}$,

$$(-X) \oplus X = (\cos\phi, \sin\phi, z) \oplus (\cos(-\phi), \sin(-\phi), -z) = (\cos(\phi + (-\phi)), \sin(\phi + (-\phi)), z + (-z)) = (1, 0, 0) \doteq E\,.$$

By commutativity (c2), $(-X)$ is clearly also a right antivector: $X \oplus (-X) = E$. In terms of original coordinates $X = (x, y, z)$ subject to the constraint in $(\star)$ we have $(-X) = (x, -y, -z)$. (A direct proof based on $(\star)$, i.e. not using the polar representation, is equally straightforward.)

Next we aim to verify the vector space axioms involving $\otimes$.

Conjecture. For any $X \in \mathrm{H}$ and $\lambda \in \mathbb{R}$ there exists a $\Lambda \in \mathrm{H}$ such that

$$\lambda \otimes X = \Lambda \oplus X\,.$$

(5) **d.** Prove this conjecture by constructing the explicit form of $\Lambda \in \mathrm{H}$ given $\lambda \in \mathbb{R}$.

Take $X = (x, y, z) = (\cos\phi, \sin\phi, z) \in \mathrm{H}$. Set $\Lambda \doteq (\cos(2\pi\lambda), \sin(2\pi\lambda), \lambda)$, then, using $(\bullet)$, a direct verification shows that

$$\Lambda \oplus X \doteq (\cos(2\pi\lambda), \sin(2\pi\lambda), \lambda) \oplus (\cos\phi, \sin\phi, z) = (\cos(2\pi\lambda + \phi), \sin(2\pi\lambda + \phi), \lambda + z)) \overset{\bullet}{=} \\ (\cos\phi\cos(2\pi\lambda) - \sin\phi\sin(2\pi\lambda), \sin\phi\cos(2\pi\lambda) + \cos\phi\sin(2\pi\lambda), z + \lambda) \doteq \lambda \otimes (\cos\phi, \sin\phi, z) \doteq \lambda \otimes X\,.$$

(5) **e.** Show that H is *not* a vector space by showing that $\otimes$ violates the axioms for scalar multiplication. (*Hint:* The conjecture may be helpful.)

We only need to disprove one of the four axioms involving scalar multiplication $\otimes$. Consider e.g.

$$\lambda \otimes (X \oplus U) = \Lambda \oplus (X \oplus U) \overset{\text{c1}}{=} (\Lambda \oplus X) \oplus U = (\lambda \otimes X) \oplus U \neq (\lambda \otimes X) \oplus (\lambda \otimes U)\,.$$

Alternatively, if $\lambda = 1$, then $\Lambda = (1, 0, 1) = (\cos 0, \sin 0, 1)$, so for $X \doteq (\cos\phi, \sin\phi, z)$ as before, we have

$$1 \otimes X \overset{\text{d}}{=} (\cos 0, \sin 0, 1) \oplus (\cos\phi, \sin\phi, z) \overset{\bullet}{=} (\cos\phi, \sin\phi, z + 1) \neq (\cos\phi, \sin\phi, z) \doteq X\,,$$

violating another basic axiom. Other axioms involving $\otimes$ may likewise be considered to disprove a vector space structure.

♣

(15) 2. Inner Product

For $v, w \in \mathbb{R}^n$, endowed with the standard vector space structure, we wish to define a real inner product

$$(\dagger) \qquad \langle v | w \rangle \doteq v^{\mathsf{T}}\, \mathrm{G}\, w\,,$$

in which, in terms of standard vector-matrix notation, with real entries v_i, g_{ij} and w_j, $1 \leq i, j \leq n$,

$$v^{\mathsf{T}} \doteq \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}\,, \qquad \mathrm{G} \doteq \begin{pmatrix} g_{11} & \dots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \dots & g_{nn} \end{pmatrix}\,, \qquad w \doteq \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\,.$$

The following theorems may be used without proof.

Jacobi's Theorem. Any symmetric matrix A can be transformed into a diagonal form $\mathrm{D} \doteq \mathrm{S}^{\mathsf{T}}\mathrm{A}\mathrm{S}$ by a suitable choice of square matrix S, in which each of the diagonal elements of D is either ± 1 or 0.

Sylvester's Law of Inertia. Recall Jacobi's Theorem. The signature (n_0, n_+, n_-), in which n_0 denotes the number of 0's and $n_\pm$ the number of ± 1's on the diagonal of D, is the same for any choice of S.

a. Use the axioms of a real inner product to infer the constraints on the matrix G, proceeding as follows.

(5) **a1.** Show that, regardless the choice of G, the definition $(\dagger)$ is consistent with the bilinearity axiom.

(5) **a2.** Find the constraint on G (or, equivalently, on its entries g_{ij}) induced by the symmetry axiom.

(5) **a3.** Likewise for the positivity and nondegeneracy axiom: $\langle v|v\rangle > 0$ for all *nonzero* vectors $v \in \mathbb{R}^n$.

Bilinearity is evident and does not constrain g_{ij}:

- $\langle \lambda u + \mu v|w\rangle \doteq g_{ij}(\lambda u + \mu v)^i w^j = g_{ij}(\lambda u^i + \mu v^i)w^j = \lambda g_{ij} u^i w^j + \mu g_{ij} v^i w^j \doteq \lambda\langle u|w\rangle + \mu\langle v|w\rangle$.
- Symmetry (next axiom) implies bilinearity, without constraints on g_{ij}.

The symmetry axiom does impose a constraint:

- $\langle v|w\rangle \doteq g_{ij} v^i w^j \overset{\star}{=} g_{ji} v^j w^i$, which equals $\langle w|v\rangle \doteq g_{ij} w^i v^j$ iff $g_{ij} = g_{ji}$, i.e. $G = G^T$ must be symmetric.

Finally, the positivity and nondegeneracy axioms pose further constraints, viz.

- $\langle v|v\rangle \doteq g_{ij} v^i v^j \geq 0$ iff the coefficient matrix G in this quadratic form has positive eigenvalues only.

♣

(20) 3. Distribution Theory

We consider a travelling wave in the form of a function $u : \mathbb{R}^2 \to \mathbb{R} : (x,t) \mapsto u(x,t) \doteq f(x-ct)$, in which $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$ is a univariate function.

(5) **a.** Show that if $f \in C^1(\mathbb{R})$, then $u \in C^1(\mathbb{R}^2)$ satisfies the following initial value problem:

$$(\star) \quad \begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{1}{c}\dfrac{\partial u}{\partial t} = 0 & \text{for } (x,t) \in \mathbb{R}^2, \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

The chain rule yields $\partial_x u(x,t) = f'(x-ct)\partial_x(x-ct) = f'(x-ct)$, respectively $\partial_t u(x,t) = f'(x-ct)\partial_t(x-ct) = -cf'(x-ct)$, whence $\partial_x u + \dfrac{1}{c}\partial_t u = 0$.

(5) **b.** Show that if $\phi \in \mathscr{S}(\mathbb{R})$, then $\displaystyle\int_{-\infty}^{\infty} \frac{d\phi(x)}{dx}\, dx = 0$.

Straightforward integration yields $\displaystyle\int_{-\infty}^{\infty} \phi'(x)dx = [\phi(x)]_{x\to-\infty}^{x\to\infty} = 0$ by definition of a rapid decay test function $\phi \in \mathscr{S}(\mathbb{R})$.

(10) **c.** Show that if $f \in \mathscr{P}(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R})$, then $u \in \mathscr{S}'(\mathbb{R}^2)$ satisfies $(\star)$ in distributional sense.
(*Hint:* Do *not* assume $f \in C^1(\mathbb{R})$. Consider a change of variables $y = x - ct$ for any fixed t.)

Consider the distributional form of $(\star)$:

$$(\star\star) \quad \int_{\mathbb{R}^2} u(x,t)\left(\partial_x\phi(x,t) + \frac{1}{c}\partial_t\phi(x,t)\right) dtdx = 0\,.$$

The change of variables

$$(*) \quad \begin{cases} y &= x - ct \\ s &= t \end{cases}$$

with inverse

$$(**) \quad \begin{cases} x &= y + cs \\ t &= s \end{cases}$$

induces a Jacobian

$$J \doteq \begin{pmatrix} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial s} \\ \frac{\partial t}{\partial y} & \frac{\partial t}{\partial s} \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

with unit determinant $|\det J| = 1$, whence $(\star\star)$ can be transformed into

$$\int_{\mathbb{R}^2} \tilde{u}(y,s) \left(\partial_y \tilde{\phi}(y,s) + \frac{1}{c} \left[-c\partial_y + \partial_s \right] \tilde{\phi}(y,s) \right) dsdy = \frac{1}{c} \int_{\mathbb{R}^2} \tilde{u}(y,s) \partial_s \tilde{\phi}(y,s) \, dsdy = \frac{1}{c} \int_{\mathbb{R}^2} f(y) \partial_s \tilde{\phi}(y,s) \, dsdy = 0\,,$$

in which $\tilde{u}(y,s) \doteq u(x,t)$ and $\tilde{\phi}(y,s) \doteq \phi(x,t)$ given the relation $(*)$. In the second last step we have used the observation that $\tilde{u}(y,s) \stackrel{**}{=} u(y+cs,s) = f(y)$ is independent of s, so that the final step follows by virtue of the result in b applied to the innermost s-integral.

♣

(**30**) **4.** FOURIER TRANSFORMATION (EXAM JANUARY 17, 2011, PROBLEM 4)

The Fourier convention used in this problem for functions of one variable is as follows:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx \quad \text{whence} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \widehat{f}(\omega)\, d\omega\,.$$

We indicate the Fourier transform of a function f by $\mathscr{F}(f)$, and the inverse Fourier transform of a function $\widehat{f}$ by $\mathscr{F}^{-1}(\widehat{f})$.

You may use the following standard limit, in which $z \in \mathbb{C}$ with real part $\operatorname{Re} z \in \mathbb{R}$:

$$\lim_{\operatorname{Re} z \to -\infty} e^z = 0\,.$$

(5) **a.** Let $\widehat{f}^+$ and $\widehat{f}^-$ be any pair of $\mathbb{C}$-valued functions defined in Fourier space, such that $\widehat{f}^-(\omega) = \widehat{f}^+(-\omega)$. Assuming that the Fourier inverses $f^\pm = \mathscr{F}^{-1}(\widehat{f}^\pm)$ exist, show that $f^-(x) = f^+(-x)$.

$f^-(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \widehat{f}^-(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \widehat{f}^+(-\omega)\, d\omega \stackrel{*}{=} -\frac{1}{2\pi} \int_{\infty}^{-\infty} e^{-i\omega' x} \widehat{f}^+(\omega')\, d\omega' = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \widehat{f}^+(\omega)\, d\omega = f^+(-x)$. In $*$ a new variable $\omega' = -\omega$ has been introduced, all other equalities follow from the given definitions.

We now consider the following particular instances:

$$\widehat{f}_s^+(\omega) = \begin{cases} e^{-s\omega} & \text{if } \omega > 0 \\ \frac{1}{2} & \text{if } \omega = 0 \\ 0 & \text{if } \omega < 0 \end{cases} \qquad (\star)$$

and $\widehat{f}_s^-(\omega) = \widehat{f}_s^+(-\omega)$, in which $s > 0$ is a parameter.

(5) **b.** Give the explicit definition of $\widehat{f}_s^-(\omega)$ in a form similar to that of $\widehat{f}_s^+(\omega)$ in Eq. $(\star)$.

Replacing all instances of ω in Eq. ($\star$) by $-\omega$ leads to

$$\widehat{f_s^-}(\omega) = \begin{cases} e^{s\omega} & \text{if } \omega < 0 \\ \frac{1}{2} & \text{if } \omega = 0 \\ 0 & \text{if } \omega > 0 \end{cases}$$

(5) **c1.** Compute $f_s^+(x) = \left(\mathscr{F}^{-1}(\widehat{f_s^+})\right)(x)$.

We have $f_s^+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\,\widehat{f_s^+}(\omega)\,d\omega = \frac{1}{2\pi}\int_0^{\infty} e^{\omega(ix-s)}\,d\omega = \frac{1}{2\pi}\frac{e^{\omega(ix-s)}}{ix-s}\Big|_0^{\infty} = \frac{1}{2\pi}\frac{1}{s-ix}$. In the last step we have used the standard limit for the complex exponential function stated above.

(5) **c2.** Compute $f_s^-(x) = \left(\mathscr{F}^{-1}(\widehat{f_s^-})\right)(x)$.

According to the result under a1 we have $f_s^-(x) = f_s^+(-x) = \frac{1}{2\pi}\frac{1}{s+ix}$.

(5) **d.** We define $\widehat{f_s} = \widehat{f_s^+} + \widehat{f_s^-}$. Give the explicit form of $\widehat{f_s}(\omega)$ and compute $f_s(x) = \left(\mathscr{F}^{-1}(\widehat{f_s})\right)(x)$.

Since $\mathscr{F}^{-1}$ is a linear operator we have $f_s = \mathscr{F}^{-1}(\widehat{f_s}) = \mathscr{F}^{-1}(\widehat{f_s^+} + \widehat{f_s^-}) = \mathscr{F}^{-1}(\widehat{f_s^+}) + \mathscr{F}^{-1}(\widehat{f_s^-}) = f_s^+ + f_s^-$. That is, $f_s(x) = \frac{1}{2\pi}\frac{1}{s-ix} + \frac{1}{2\pi}\frac{1}{s+ix} = \frac{1}{\pi}\frac{s}{x^2+s^2}$.

(5) **e.** Show that $\mathscr{F}(f_s * f_t) = \widehat{f}_{s+t}$.

We have $\mathscr{F}(f_s * f_t) \stackrel{*}{=} \mathscr{F}(f_s)\,\mathscr{F}(f_t) = \widehat{f}_s\,\widehat{f}_t \stackrel{\star}{=} \widehat{f}_{s+t}$. In $*$ we have used a well-known Fourier theorem, whereas $\star$ makes explicit use of the property $\widehat{f}_s(\omega) = e^{-s|\omega|}$.

THE END