EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10. Date: Friday January 24, 2020. Time: 09:00–12:00. Place: Vertigo 4.06 A.

READ THIS FIRST!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- The use of course notes is allowed, provided it is in immaculate state. The use of other notes, calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- Motivate your answers. You may provide your answers in Dutch or English.

GOOD LUCK!

(**35**) 1. VECTOR SPACE

We consider the subset H of points in $\mathbb{R}^3$ given by

$$(\star) \qquad \mathrm{H} \doteq \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2+y^2=1 \right\} ,$$

equipped with internal and external operations $\oplus : \mathrm{H}\times\mathrm{H} \to \mathrm{H}$, respectively $\otimes : \mathbb{R}\times\mathrm{H} \to \mathrm{H}$, viz.

$$\begin{aligned} (x,y,z) \oplus (u,v,w) &\doteq (xu-yv, yu+xv, z+w) , \\ \lambda \otimes (x,y,z) &\doteq (x\cos(2\pi\lambda) - y\sin(2\pi\lambda), y\cos(2\pi\lambda) + x\sin(2\pi\lambda), z+\lambda) . \end{aligned}$$

For notational convenience we abbreviate elements of H as $X \doteq (x,y,z)$, $U \doteq (u,v,w)$ et cetera.

(5) **a.** Prove closure, i.e. show that $X \oplus U \in \mathrm{H}$ and $\lambda \otimes X \in \mathrm{H}$ for all $X, U \in \mathrm{H}$ and $\lambda \in \mathbb{R}$.

(5) **b.** Show that $X \in \mathrm{H}$ can be parametrized such that the constraint in $(\star)$ is automatically fulfilled.
(*Hint:* Introduce an angle $\phi \in \mathbb{R}$ and consider polar coordinates for the (x,y)-plane.)

We now investigate whether $(\star)$, furnished with the operators $\oplus$ and $\otimes$, satisfies all vector space axioms. We consider the abelian group requirement for $\oplus$ first.

You may use the following lemma.

Lemma. For $\phi, \theta \in \mathbb{R}$ we have

$$\begin{aligned} \cos(\phi \pm \theta) &= \cos\phi\,\cos\theta \mp \sin\phi\,\sin\theta , \\ \sin(\phi \pm \theta) &= \sin\phi\,\cos\theta \pm \cos\phi\,\sin\theta . \end{aligned}$$

(5) **c1.** Prove associativity: $(X \oplus U) \oplus A = X \oplus (U \oplus A)$ for all $X, U, A \in \mathrm{H}$.
(*Hint:* Exploit your observation in b and use the lemma.)

($2\frac{1}{2}$) **c2.** Prove commutativity: $X \oplus U = U \oplus X$ for all $X, U \in$ H.

($2\frac{1}{2}$) **c3.** Show that $E \doteq (1, 0, 0) \in$ H is the neutral element for $\oplus$.

(5) **c4.** State the explicit form of the antivector $(-X) \in$ H for any given $X \in$ H, and prove $(-X) \oplus X = E$.

Next we aim to verify the vector space axioms involving $\otimes$.

Conjecture. For any $X \in$ H and $\lambda \in \mathbb{R}$ there exists a $\Lambda \in$ H such that

$$\lambda \otimes X = \Lambda \oplus X \, .$$

(5) **d.** Prove this conjecture by constructing the explicit form of $\Lambda \in$ H given $\lambda \in \mathbb{R}$.

(5) **e.** Show that H is *not* a vector space by showing that $\otimes$ violates the axioms for scalar multiplication. (*Hint:* The conjecture may be helpful.)

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(**15**) 2. Inner Product

For $v, w \in \mathbb{R}^n$, endowed with the standard vector space structure, we wish to define a real inner product

$$(\dagger) \qquad \langle v|w\rangle \doteq v^{\mathsf{T}}\, \mathrm{G}\, w \, ,$$

in which, in terms of standard vector-matrix notation, with real entries v_i, g_{ij} and w_j, $1 \leq i, j \leq n$,

$$v^{\mathsf{T}} \doteq \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \, , \qquad \mathrm{G} \doteq \begin{pmatrix} g_{11} & \dots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \dots & g_{nn} \end{pmatrix} \, , \qquad w \doteq \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \, .$$

The following theorems may be used without proof.

Jacobi's Theorem. Any symmetric matrix A can be transformed into a diagonal form $\mathrm{D} \doteq \mathrm{S}^{\mathsf{T}}\mathrm{A}\mathrm{S}$ by a suitable choice of square matrix S, in which each of the diagonal elements of D is either ± 1 or 0.

Sylvester's Law of Inertia. Recall Jacobi's Theorem. The signature (n_0, n_+, n_-), in which n_0 denotes the number of 0's and $n_\pm$ the number of ± 1's on the diagonal of D, is the same for any choice of S.

a. Use the axioms of a real inner product to infer the constraints on the matrix G, proceeding as follows.

(5) **a1.** Show that, regardless the choice of G, the definition $(\dagger)$ is consistent with the bilinearity axiom.

(5) **a2.** Find the constraint on G (or, equivalently, on its entries g_{ij}) induced by the symmetry axiom.

(5) **a3.** Likewise for the positivity and nondegeneracy axiom: $\langle v|v\rangle > 0$ for all *nonzero* vectors $v \in \mathbb{R}^n$.

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(**20**) 3. DISTRIBUTION THEORY

We consider a travelling wave in the form of a function $u : \mathbb{R}^2 \to \mathbb{R} : (x,t) \mapsto u(x,t) \doteq f(x-ct)$, in which $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$ is a univariate function.

(5) **a.** Show that if $f \in C^1(\mathbb{R})$, then $u \in C^1(\mathbb{R}^2)$ satisfies the following initial value problem:

$$(\star) \quad \begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{1}{c}\dfrac{\partial u}{\partial t} &= 0 \quad \text{for } (x,t) \in \mathbb{R}^2, \\ u(x,0) &= f(x) \quad \text{for } x \in \mathbb{R}. \end{cases}$$

(5) **b.** Show that if $\phi \in \mathscr{S}(\mathbb{R})$, then $\displaystyle\int_{-\infty}^{\infty} \frac{d\phi(x)}{dx}\, dx = 0$.

(10) **c.** Show that if $f \in \mathscr{P}(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R})$, then $u \in \mathscr{S}'(\mathbb{R}^2)$ satisfies $(\star)$ in distributional sense.
(*Hint:* Do *not* assume $f \in C^1(\mathbb{R})$. Consider a change of variables $y = x - ct$ for any fixed t.)

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(**30**) 4. FOURIER TRANSFORMATION (EXAM JANUARY 17, 2011, PROBLEM 4)

The Fourier convention used in this problem for functions of one variable is as follows:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx \quad \text{whence} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, \widehat{f}(\omega)\, d\omega\,.$$

We indicate the Fourier transform of a function f by $\mathscr{F}(f)$, and the inverse Fourier transform of a function $\widehat{f}$ by $\mathscr{F}^{-1}(\widehat{f})$.

You may use the following standard limit, in which $z \in \mathbb{C}$ with real part $\mathrm{Re}\, z \in \mathbb{R}$:

$$\lim_{\mathrm{Re}\, z \to -\infty} e^z = 0\,.$$

(5) **a.** Let $\widehat{f}^+$ and $\widehat{f}^-$ be any pair of $\mathbb{C}$-valued functions defined in Fourier space, such that $\widehat{f}^-(\omega) = \widehat{f}^+(-\omega)$. Assuming that the Fourier inverses $f^\pm = \mathscr{F}^{-1}(\widehat{f}^\pm)$ exist, show that $f^-(x) = f^+(-x)$.

We now consider the following particular instances:

$$\widehat{f}_s^+(\omega) = \begin{cases} e^{-s\omega} & \text{if } \omega > 0 \\ \frac{1}{2} & \text{if } \omega = 0 \\ 0 & \text{if } \omega < 0 \end{cases} \qquad (\star)$$

and $\widehat{f}_s^-(\omega) = \widehat{f}_s^+(-\omega)$, in which $s > 0$ is a parameter.

(5) **b.** Give the explicit definition of $\widehat{f}_s^-(\omega)$ in a form similar to that of $\widehat{f}_s^+(\omega)$ in Eq. $(\star)$.

(5) **c1.** Compute $f_s^+(x) = \left(\mathscr{F}^{-1}(\widehat{f}_s^+)\right)(x)$.

(5) **c2.** Compute $f_s^-(x) = \left(\mathscr{F}^{-1}(\widehat{f_s^-})\right)(x)$.

(5) **d.** We define $\widehat{f_s} = \widehat{f_s^+} + \widehat{f_s^-}$. Give the explicit form of $\widehat{f_s}(\omega)$ and compute $f_s(x) = \left(\mathscr{F}^{-1}(\widehat{f_s})\right)(x)$.

(5) **e.** Show that $\mathscr{F}\left(f_s * f_t\right) = \widehat{f_{s+t}}$.

THE END