EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10. Date: Friday January 25, 2019. Time: 09:00–12:00. Place: AUD 13.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- The use of course notes is allowed, provided it is in immaculate state. The use of other notes, calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- Motivate your answers. You may provide your answers in Dutch or English.

GOOD LUCK!

(**30**) **1.** GROUP HOMOMORPHISMS

(5) **a1.** Show that $\mathbb{Z}$, equipped with default integer addition, constitutes a group.

Closure: The sum of two integers is an integer. Verification of group axioms:

- Associativity: For all $z_1, z_2, z_3 \in \mathbb{Z}$ we have $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \doteq z_1 + z_2 + z_3$.
- Identity: For all $z \in \mathbb{Z}$ we have $z + 0 = 0 + z = z$, so $0 \in \mathbb{Z}$ is the identity element.
- Inverse: For all $z \in \mathbb{Z}$ we have $z + (-z) = (-z) + z = 0$, so $-z \in \mathbb{Z}$ is the inverse element of z.

(5) **a2.** Show that $\mathbb{C}\backslash\{0\}$, equipped with default complex multiplication, constitutes a group.

Closure: The product of two nonzero complex numbers is a nonzero complex number. Verification of group axioms:

- Associativity: For all $z_1, z_2, z_3 \in \mathbb{C}\backslash\{0\}$ we have $(z_1 z_2) z_3 = z_1 (z_2 z_3) \doteq z_1 z_2 z_3$.
- Identity: For all $z \in \mathbb{C}\backslash\{0\}$ we have $z \times 1 = 1 \times z = z$, so $1 \in \mathbb{C}\backslash\{0\}$ is the identity element.
- Inverse: For all $z \in \mathbb{C}\backslash\{0\}$ we have $z(1/z) = (1/z)z = 1$, so $1/z \in \mathbb{Z}$ is the inverse element of z. Note that if $z = a + ib$ for some $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$, then $1/z = (a - ib)/(a^2 + b^2)$.

(5) **a3.** Show that the set $\mathbb{S} \doteq \{1, -1, i, -i\}$ constitutes a finite group under the same operation as in a2.

Observe that $\mathbb{S} \subset \mathbb{C}\backslash\{0\}$ is closed under multiplication. Furthermore, for each $u \in \mathbb{S}$ it is easy to see that $u^{-1} \in \mathbb{S}$ as well, e.g. by constructing the group multiplication table for $\mathbb{S}$. Consequently, $\mathbb{S} \subset \mathbb{C}\backslash\{0\}$ is a subgroup.

×	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

Definition. A *group homomorphism* between two groups $\{G, \circ\}$ and $\{H, \bullet\}$ is a mapping

$$\phi : G \to H : g \mapsto \phi(g) \quad \text{such that} \quad \phi(g_1 \circ g_2) = \phi(g_1) \bullet \phi(g_2) .$$

By $e_G \in G$, $e_H \in H$ we denote the unit elements of G and H; $g^{-1} \in G$, $h^{-1} \in H$ denote the inverses of $g \in G$ and $h \in H$.

($2\frac{1}{2}$) **b1.** Show that $\phi(e_G) = e_H$.

Suppose $h = \phi(g) \in \phi(G) \subset H$, then $h = \phi(g) = \phi(g \circ e_G) = \phi(g) \bullet \phi(e_G) = h \bullet \phi(e_G)$, whence $\phi(e_G) = e_{\phi(G)} = e_H$. The last identity makes use of the fact that the identity element of a group (H in this case) equals that of any of its subgroups (here $\phi(G) \subset H$). To see this, multiply the identity $\phi(g) \bullet e_H = \phi(g)$ for all $g \in G$, which holds on $\phi(G) \subset H$, from the left with $h \bullet \phi(g)^{-1}$, in which $h \in H$ is arbitrary. This yields $(h \bullet \phi(g)^{-1}) \bullet (\phi(g) \bullet e_H) = h \bullet e_H = h$, which holds on $H \supset \phi(G)$.

($2\frac{1}{2}$) **b2.** Show that $\phi(g^{-1}) = \phi(g)^{-1}$.

Consider $e_H = \phi(e_G) = \phi(g \circ g^{-1}) = \phi(g) \bullet \phi(g^{-1})$, which shows that $\phi(g^{-1}) = \phi(g)^{-1}$.

(5) **c.** Show that $\psi : \mathbb{Z} \to \mathbb{S} : n \mapsto \psi(n) \doteq i^n$ is a group homomorphism.

We have $\psi(n_1 + n_2) = i^{n_1+n_2} = i^{n_1} i^{n_2} = \psi(n_1)\psi(n_2)$.

Definition. The *kernel* of ϕ is $\text{Ker}\,\phi = \{g \in G \,|\, \phi(g) = e_H\}$. The *image* of ϕ is $\text{Im}\,\phi = \{\phi(g) \in H \,|\, g \in G\}$.

Definition. A group homomorphism $\phi : G \to H$ is an *epimorphism* if it is surjective, i.e. if $\text{Im}\,\phi = H$, and a *monomorphism* if it is injective, i.e. if $\text{Ker}\,\phi = \{e_G\}$.

($2\frac{1}{2}$) **d1.** Recall **c**. Specify $\text{Ker}\,\psi$. Is ψ a monomorphism?

Since $i^n = 1$ implies $n = 4k$ for some $k \in \mathbb{Z}$ we have $\text{Ker}\,\psi = \{4k \,|\, k \in \mathbb{Z}\}$. Thus ψ is not a monomorphism (unless one considers its arguments modulo 4).

($2\frac{1}{2}$) **d2.** Recall **c**. Specify $\text{Im}\,\psi$. Is ψ an epimorphism?

Since $i^n = i^{n \bmod 4}$ it suffices to consider $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$ for some $k \in \mathbb{Z}$. That is, $\text{Im}\,\psi = \{1, -1, i, -i\} \doteq \mathbb{S}$. With the specified codomain ψ is therefore indeed an epimorphism.

As an aside, $\psi : \mathbb{Z}\backslash 4\mathbb{Z} \to \mathbb{S} : n \mapsto \psi(n) \doteq i^n$ is both a monomorphism as well as an epimorphism, i.e. an isomorphism. The notation $\mathbb{Z}\backslash 4\mathbb{Z}$ is the set of equivalence classes $[n]$ determined by any integer $n \in \mathbb{Z}$, in which we identify $[n] = [m]$, or $n \equiv m$, for two integers $n, m \in \mathbb{Z}$ if $n = m \mod 4$, i.e. if the difference $n - m$ is a multiple of 4.

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(**20**) 2. Linear Operator

Consider a real vector space V with basis $\{e_1, e_2\}$, and an operator $T : V \times V \to \mathbb{R} : (v, w) \mapsto T(v, w)$ with the following properties:

- T is bilinear;

- $T(v, w) = -T(w, v)$;
- $T(e_1, e_2) = 1$.

Define $v = \sum_{i=1}^{2} v^i e_i$ and $w = \sum_{i=1}^{2} w^i e_i$, with $v^i, w^i \in \mathbb{R}$, $i, j = 1, 2$.

(10) **a.** Show that $T(v, w) = \sum_{i=1}^{2}\sum_{j=1}^{2} \epsilon_{ij} v^i w^j$ for certain coefficients ϵ_{ij} and compute their values.

Inserting $v = \sum_{i=1}^{2} v^i e_i$, $w = \sum_{i=1}^{2} w^i e_i$ into $T(v, w)$ yields $T(\sum_{i=1}^{2} v^i e_i, \sum_{j=1}^{2} w^j e_j) = \sum_{i=1}^{2}\sum_{j=1}^{2} T(e_i, e_j) v^i w^j$, in which we have used bilinearity in the last step. Apparently $\epsilon_{ij} = T(e_i, e_j)$. This proves the existence of the stipulated coefficients. As for their values, note that $\epsilon_{ij} = T(e_i, e_j) = -T(e_j, e_i) = -\epsilon_{ji}$, whence $\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{12} = -\epsilon_{21} = T(e_1, e_2) = 1$.

Let $\mathbb{M}_n$ denote the linear space of $n \times n$ square matrices A with entries $A_{ij} \in \mathbb{R}$, $1 \le i, j \le n$. Consider the map $\mathscr{S}_\lambda : \mathbb{M}_n \to \mathbb{M}_n : A \mapsto \mathscr{S}_\lambda(A)$, in which $\lambda \in \mathbb{R}$ is a parameter, given by

$$\left(\mathscr{S}_\lambda(A)\right)_{ij} = \lambda \left(A_{ij} + A_{ji}\right) .$$

We furthermore define the standard inner product for $A, B \in \mathbb{M}_n$ as follows:

$$\langle A|B\rangle = \text{trace}\,(AB^T) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} B_{ij} .$$

We call a linear operator $L \in \mathscr{L}(V, V)$ on a real inner product space V symmetric if $\langle Lv|w\rangle = \langle v|Lw\rangle$ for all $v, w \in V$. We call L a projection if $L \circ L = L$, meaning $L(L(v)) = L(v)$ for all $v \in V$.

(5) **b1.** Show that $\mathscr{S}_\lambda \in \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ is symmetric for any $\lambda \in \mathbb{R}$.

Consider

$$\begin{aligned}
\langle \mathscr{S}_\lambda(A)|B\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\mathscr{S}_\lambda(A)\right)_{ij} B_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\lambda \left(A_{ij} + A_{ji}\right)\right) B_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda A_{ij} B_{ij} + \lambda A_{ji} B_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda A_{ij} B_{ij} + \lambda A_{ij} B_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} \left(\lambda \left(B_{ij} + B_{ji}\right)\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} \left(\mathscr{S}_\lambda(B)\right)_{ij} = \langle A|\mathscr{S}_\lambda(B)\rangle .
\end{aligned}$$

(5) **b2.** For which $\lambda \in \mathbb{R}$ does $\mathscr{S}_\lambda \in \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ define a projection? Prove your answer.

Note that, for arbitrary $A \in \mathbb{M}_n$,

$$\left(\mathscr{S}_\lambda(\mathscr{S}_\lambda(A))\right)_{ij} = \lambda \left(\mathscr{S}_\lambda(A)_{ij} + \mathscr{S}_\lambda(A)_{ji}\right) = 2\lambda^2 \left(A_{ij} + A_{ji}\right) = 2\lambda \mathscr{S}_\lambda(A)_{ij} .$$

By definition of a projection we must therefore impose either $2\lambda = 1$, whence $\lambda = 1/2$, or the trivial case, $\lambda = 0$, corresponding to the trivial map $\mathscr{S}_0 = 0 \in \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$.

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(25) 3. PDE Theory and Fourier Analysis (Exam February 1, 2017, Problem 3)

The so-called Bloch-Torrey equations describe the evolution of the 3 components of the magnetization vector field $\vec{M}(x,y,z,t) = (M_x(x,y,z,t), M_y(x,y,z,t), M_z(x,y,z,t))$ induced in a patient placed in an MRI scanner with static magnetic field $\vec{B}_0 = (0,0,B_0)$. In particular, the $\mathbb{C}$-valued transversal magnetization $m(x,y,z,t) \doteq M_x(x,y,z,t) + iM_y(x,y,z,t)$ satisfies the partial differential equation

$$\frac{\partial m}{\partial t} = -i\omega_0 m - \frac{m}{T_2} + D\Delta m\,,$$

in which $\Delta \doteq \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian. The so-called Larmor frequency $\omega_0 > 0$ is a constant proportional to B_0. We likewise assume $D > 0$, the diffusion coefficient, and $T_2 > 0$, the spin-spin relaxation time, to be constant.

(5) **a.** Give the corresponding evolution equation for $\widehat{m}(\omega_x,\omega_y,\omega_z,t)$ in the spatial Fourier domain.

We have

$$\frac{\partial \widehat{m}}{\partial t} = -\left(i\omega_0 + \frac{1}{T_2} + D\|\omega\|^2\right)\widehat{m}\,,$$

in which $\|\omega\|^2 \doteq \omega_x^2 + \omega_y^2 + \omega_z^2$.

At the start of the scan sequence, the system is initialized so that $m(x,y,z,t\!=\!0) = m_0(x,y,z)$, with Fourier transform $\widehat{m}_0(\omega_x,\omega_y,\omega_z)$.

(5) **b.** Determine $\widehat{m}(\omega_x,\omega_y,\omega_z,t)$ as a function of time $t \geq 0$, given $\widehat{m}_0(\omega_x,\omega_y,\omega_z)$.

We have

$$\widehat{m}(\omega_x,\omega_y,\omega_z,t) = \widehat{m}_0(\omega_x,\omega_y,\omega_z)\,e^{-(i\omega_0+1/T_2+D\|\omega\|^2)t}\,.$$

c. Show that $\mu(t) \doteq \int_{\mathbb{R}^3} m(x,y,z,t)dxdydz$ is not preserved as a function of time by proving the following statements:

($2\frac{1}{2}$) **c1.** $|\mu(t)|$ decays exponentially over time towards zero.

($2\frac{1}{2}$) **c2.** $\mu(t)/|\mu(t)|$ rotates clockwise with uniform angular velocity around the origin of the $\mathbb{C}$-plane.

We have

$$\mu(t) = \int_{\mathbb{R}^3} m(x,y,z,t)dxdydz = \widehat{m}(0,0,0,t) = \widehat{m}_0(0,0,0)\,e^{-(i\omega_0+1/T_2)t}\,.$$

Both statements follow by inspection: $|\mu(t)| = |\widehat{m}_0(0,0,0)|\,e^{-t/T_2}$, resp. $\mu(t)/|\mu(t)| = e^{-i(\omega_0 t-\phi)}$, in which the phase angle ϕ is defined such that $e^{i\phi} \stackrel{\text{def}}{=} \widehat{m}_0(0,0,0)/|\widehat{m}_0(0,0,0)|$.

(10) **d.** Determine $m(x,y,z,t)$ as a function of time $t \geq 0$, given $m_0(x,y,z)$.

We have the solution implicitly in Fourier space, cf. problem b. Write it as

$$\widehat{m}(\omega_x,\omega_y,\omega_z,t) = e^{-(i\omega_0+1/T_2)t}\,\widehat{m}_0(\omega_x,\omega_y,\omega_z)\,\widehat{\phi}_t(\omega_x,\omega_y,\omega_z)\,,$$

in which

$$\widehat{\phi}_t(\omega_x,\omega_y,\omega_z) = e^{-D\|\omega\|^2 t}\,.$$

In order to obtain its Fourier inverse we may apply one of the convolution theorems (the overall factor $e^{-(i\omega_0+1/T_2)t}$ can be seen as a constant factor, since it does not involve $\omega = (\omega_x,\omega_y,\omega_z)$, and can be separated by virtue of linearity of the Fourier transform):

$$m(x,y,z,t) = e^{-(i\omega_0+1/T_2)t}\,\mathscr{F}^{-1}\left(\widehat{m}_0\,\widehat{\phi}_t\right)(x,y,z) = e^{-(i\omega_0+1/T_2)t}\,(m_0 * \phi_t)(x,y,z)\,,$$

in which

$$\phi_t(x,y,z) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-D\|\omega\|^2 t + i\omega\cdot x} d\omega = \frac{1}{\sqrt{4\pi D t}^3}\, e^{-\frac{x^2+y^2+z^2}{4DT}} \,.$$

Here we have abbreviated $\omega \cdot x = \omega_x x + \omega_y y + \omega_z z$.

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(25) 4. Distribution Theory & Scale Space

Consider the discontinuous function sign : $\mathbb{R} \to \mathbb{R} : x \mapsto \text{sign}\,(x)$, given by

$$(\star) \qquad \text{sign}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}$$

Below we consider its derivative sign$'$ in the sense of distribution theory, respectively scale space theory.

(10) **a.** Show that, in distributional sense, $\text{sign}'(x) = 2\delta(x)$, in which $\delta \in \mathscr{S}'(\mathbb{R})$ is the Dirac function.

Hint: Use the proper definition for the regular tempered distribution sign $\in \mathscr{S}'(\mathbb{R})$ associated with $(\star)$.

For any $\phi \in \mathscr{S}(\mathbb{R})$ we have $\text{sign}'(\phi) = -\text{sign}(\phi') = -\int_{-\infty}^{\infty} \text{sign}(y)\,\phi'(y)\,dy = \int_{-\infty}^{0} \phi'(y)\,dy - \int_0^\infty \phi'(y)\,dy = 2\phi(0) = 2\delta(\phi)$, whence $\text{sign}' = 2\delta \in \mathscr{S}'(\mathbb{R})$. Although this is not a regular tempered distribution we may say, by notational convention, that it corresponds to the formal 'function-under-the-integral' identity $\text{sign}'(x) = 2\delta(x)$.

The scale space representation of $f \in \mathscr{S}'(\mathbb{R})$ is the scale-parametrized function $f_\sigma \in C^\infty(\mathbb{R}) \cap \mathscr{S}'(\mathbb{R})$ defined by the convolution product $f_\sigma = f * \phi_\sigma$, with $\sigma \in \mathbb{R}^+$ and

$$\phi_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right) .$$

(10) **b.** Show that, in the sense of scale space theory, $\text{sign}'_\sigma(x) = 2\phi_\sigma(x)$.

Evaluating the convolution integral $(\text{sign} * \phi_\sigma)'(x)$ boils down to substituting the generic test function $\phi \in \mathscr{S}(\mathbb{R})$ under a by a 2-parameter Gaussian, viz.

$$\phi(y) \to \phi_{x,\sigma}(y) = \phi_\sigma(x-y)\,,$$

which, using the distributional identity obtained under a, immediately produces $\text{sign}'_\sigma(x) = 2\delta(\phi_{x,\sigma}) = 2\phi_\sigma(x)$.

Lemma. Independent of $\sigma \in \mathbb{R}^+$ we have $\displaystyle\int_{-\infty}^{\infty} \phi_\sigma(x)\,dx = 1$.

(5) **c.** Prove: $\displaystyle\int_{-\infty}^{\infty} x^n\,\phi_\sigma^{(n)}(x)\,dx = (-1)^n n!$ for all $\sigma \in \mathbb{R}^+$, in which $\phi_\sigma^{(n)}(x) = \dfrac{d^n\phi_\sigma(x)}{dx^n}$.

Proof by induction. For $n = 0$ one obtains, after substituting $y = \sigma x$, the standard integral $\int_{-\infty}^{\infty} \phi_\sigma(y)\,dy = \int_{-\infty}^{\infty} \phi(x)\,dx = 1$, in which $\phi(x) = \phi_1(x)$ is the standard normalized Gaussian. Using the induction hypothesis for $n \in \mathbb{Z}_0$ a partial integration step yields

$$\int_{-\infty}^{\infty} x^{n+1}\phi_\sigma^{(n+1)}(x)\,dx = \underbrace{\left[x^{n+1}\phi_\sigma^{(n)}(x)\right]_{-\infty}^{\infty}}_{=0} - (n+1)\int_{-\infty}^{\infty} x^n\,\phi_\sigma^{(n)}(x)\,dx = -(n+1)(-1)^n n! = (-1)^{(n+1)}(n+1)!\,.$$

THE END