EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10. Date: Friday January 25, 2019. Time: 09:00–12:00. Place: AUD 13.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- The use of course notes is allowed, provided it is in immaculate state. The use of other notes, calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- Motivate your answers. You may provide your answers in Dutch or English.

GOOD LUCK!

(**30**) **1.** GROUP HOMOMORPHISMS

(5) **a1.** Show that $\mathbb{Z}$, equipped with default integer addition, constitutes a group.

(5) **a2.** Show that $\mathbb{C}\backslash\{0\}$, equipped with default complex multiplication, constitutes a group.

(5) **a3.** Show that the set $\mathbb{S} \doteq \{1, -1, i, -i\}$ constitutes a finite group under the same operation as in a2.

Definition. A *group homomorphism* between two groups $\{G, \circ\}$ and $\{H, \bullet\}$ is a mapping

$$\phi : G \to H : g \mapsto \phi(g) \quad \text{such that} \quad \phi(g_1 \circ g_2) = \phi(g_1) \bullet \phi(g_2)\,.$$

By $e_G \in G$, $e_H \in H$ we denote the unit elements of G and H; $g^{-1} \in G$, $h^{-1} \in H$ denote the inverses of $g \in G$ and $h \in H$.

($2\frac{1}{2}$) **b1.** Show that $\phi(e_G) = e_H$.

($2\frac{1}{2}$) **b2.** Show that $\phi(g^{-1}) = \phi(g)^{-1}$.

(5) **c.** Show that $\psi : \mathbb{Z} \to \mathbb{S} : n \mapsto \psi(n) \doteq i^n$ is a group homomorphism.

Definition. The *kernel* of ϕ is $\text{Ker}\,\phi = \{g \in G \mid \phi(g) = e_H\}$. The *image* of ϕ is $\text{Im}\,\phi = \{\phi(g) \in H \mid g \in G\}$.

Definition. A group homomorphism $\phi : G \to H$ is an *epimorphism* if it is surjective, i.e. if $\text{Im}\,\phi = H$, and a *monomorphism* if it is injective, i.e. if $\text{Ker}\,\phi = \{e_G\}$.

($2\frac{1}{2}$) **d1.** Recall **c**. Specify $\text{Ker}\,\psi$. Is ψ a monomorphism?

($2\frac{1}{2}$) **d2.** Recall **c**. Specify $\text{Im}\,\psi$. Is ψ an epimorphism?

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(**20**) **2.** Linear Operator

Consider a real vector space V with basis $\{e_1, e_2\}$, and an operator $T : V \times V \to \mathbb{R} : (v, w) \mapsto T(v, w)$ with the following properties:

- T is bilinear;
- $T(v, w) = -T(w, v)$;
- $T(e_1, e_2) = 1$.

Define $v = \sum_{i=1}^{2} v^i e_i$ and $w = \sum_{i=1}^{2} w^i e_i$, with $v^i, w^i \in \mathbb{R}$, $i, j = 1, 2$.

(10) **a.** Show that $T(v, w) = \sum_{i=1}^{2}\sum_{j=1}^{2} \epsilon_{ij} v^i w^j$ for certain coefficients ϵ_{ij} and compute their values.

Let $\mathbb{M}_n$ denote the linear space of $n \times n$ square matrices A with entries $A_{ij} \in \mathbb{R}$, $1 \leq i, j \leq n$. Consider the map $\mathscr{S}_\lambda : \mathbb{M}_n \to \mathbb{M}_n : A \mapsto \mathscr{S}_\lambda(A)$, in which $\lambda \in \mathbb{R}$ is a parameter, given by

$$(\mathscr{S}_\lambda(A))_{ij} = \lambda \left(A_{ij} + A_{ji}\right) .$$

We furthermore define the standard inner product for $A, B \in \mathbb{M}_n$ as follows:

$$\langle A | B \rangle = \text{trace}\,(AB^T) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} B_{ij} .$$

We call a linear operator $L \in \mathscr{L}(V, V)$ on a real inner product space V symmetric if $\langle Lv | w \rangle = \langle v | Lw \rangle$ for all $v, w \in V$. We call L a projection if $L \circ L = L$, meaning $L(L(v)) = L(v)$ for all $v \in V$.

(5) **b1.** Show that $\mathscr{S}_\lambda \in \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ is symmetric for any $\lambda \in \mathbb{R}$.

(5) **b2.** For which $\lambda \in \mathbb{R}$ does $\mathscr{S}_\lambda \in \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ define a projection? Prove your answer.

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(**25**) **3.** PDE Theory and Fourier Analysis (Exam February 1, 2017, Problem 3)

The so-called Bloch-Torrey equations describe the evolution of the 3 components of the magnetization vector field $\vec{M}(x, y, z, t) = (M_x(x, y, z, t), M_y(x, y, z, t), M_z(x, y, z, t))$ induced in a patient placed in an MRI scanner with static magnetic field $\vec{B}_0 = (0, 0, B_0)$. In particular, the $\mathbb{C}$-valued transversal magnetization $m(x, y, z, t) \doteq M_x(x, y, z, t) + iM_y(x, y, z, t)$ satisfies the partial differential equation

$$\frac{\partial m}{\partial t} = -i\omega_0 m - \frac{m}{T_2} + D\Delta m ,$$

in which $\Delta \doteq \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian. The so-called Larmor frequency $\omega_0 > 0$ is a constant proportional to B_0. We likewise assume $D > 0$, the diffusion coefficient, and $T_2 > 0$, the spin-spin relaxation time, to be constant.

(5) **a.** Give the corresponding evolution equation for $\widehat{m}(\omega_x, \omega_y, \omega_z, t)$ in the spatial Fourier domain.

At the start of the scan sequence, the system is initialized so that $m(x, y, z, t=0) = m_0(x, y, z)$, with Fourier transform $\widehat{m}_0(\omega_x, \omega_y, \omega_z)$.

(5) **b.** Determine $\widehat{m}(\omega_x, \omega_y, \omega_z, t)$ as a function of time $t \geq 0$, given $\widehat{m}_0(\omega_x, \omega_y, \omega_z)$.

c. Show that $\mu(t) \doteq \int_{\mathbb{R}^3} m(x, y, z, t) dx dy dz$ is not preserved as a function of time by proving the following statements:

($2\frac{1}{2}$) **c1.** $|\mu(t)|$ decays exponentially over time towards zero.

($2\frac{1}{2}$) **c2.** $\mu(t)/|\mu(t)|$ rotates clockwise with uniform angular velocity around the origin of the $\mathbb{C}$-plane.

(10) **d.** Determine $m(x, y, z, t)$ as a function of time $t \geq 0$, given $m_0(x, y, z)$.

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(25) 4. Distribution Theory & Scale Space

Consider the discontinuous function sign : $\mathbb{R} \to \mathbb{R} : x \mapsto \text{sign}\,(x)$, given by

$$(\star) \qquad \text{sign}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}$$

Below we consider its derivative sign$'$ in the sense of distribution theory, respectively scale space theory.

(10) **a.** Show that, in distributional sense, $\text{sign}'(x) = 2\delta(x)$, in which $\delta \in \mathscr{S}'(\mathbb{R})$ is the Dirac function.

Hint: Use the proper definition for the regular tempered distribution $\text{sign} \in \mathscr{S}'(\mathbb{R})$ associated with $(\star)$.

The scale space representation of $f \in \mathscr{S}'(\mathbb{R})$ is the scale-parametrized function $f_\sigma \in C^\infty(\mathbb{R}) \cap \mathscr{S}'(\mathbb{R})$ defined by the convolution product $f_\sigma = f * \phi_\sigma$, with $\sigma \in \mathbb{R}^+$ and

$$\phi_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right).$$

(10) **b.** Show that, in the sense of scale space theory, $\text{sign}'_\sigma(x) = 2\phi_\sigma(x)$.

Lemma. Independent of $\sigma \in \mathbb{R}^+$ we have $\displaystyle\int_{-\infty}^{\infty} \phi_\sigma(x)\, dx = 1$.

(5) **c.** Prove: $\displaystyle\int_{-\infty}^{\infty} x^n \, \phi_\sigma^{(n)}(x)\, dx = (-1)^n n!$ for all $\sigma \in \mathbb{R}^+$, in which $\phi_\sigma^{(n)}(x) = \dfrac{d^n \phi_\sigma(x)}{dx^n}$.

THE END