EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10 & 8D020. Date: Wednesday January 27 2016. Time: 13:30–16:30. Place: AUD 12.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 5 problems. The maximum credit for each item is indicated in the margin.
- The use of course notes is allowed, provided it is in immaculate state. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- Motivate your answers. You may provide your answers in Dutch or English.

GOOD LUCK!

(**25**) **1.** GROUP

Definition 1 A *group* is a collection G together with an internal operation

$$\circ : G \times G \longrightarrow G : (x, y) \mapsto x \circ y \,,$$

such that

- the operation is associative, i.e. $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in G$,
- there exists an identity element $e \in G$ such that $x \circ e = e \circ x = x$ for all $x \in G$, and
- for each $x \in G$ there exists an inverse element x^{inv} such that $x^{\text{inv}} \circ x = x \circ x^{\text{inv}} = e$.

Definition 2 Recall Definition 1. If, in addition, $x \circ y = y \circ x$ for all $x, y \in G$, then the group is called *commutative*, or *abelian*.

(10) **a.** Prove the following properties for a group G:

- The identity element $e \in G$ is unique.
- The identity element equals its own inverse: $e^{\text{inv}} = e$.
- The inverse x^{inv} of a given element $x \in G$ is unique.
- The inverse of the inverse of a given element $x \in G$ reproduces that element: $(x^{\text{inv}})^{\text{inv}} = x$.

The identity element $e \in G$ is unique: Suppose $e_1 \neq e_2$ are both identity elements in G. Consider the identity $x \circ e = e \circ x = x$, which holds for all $x \in G$, twice, once for $(e, x) = (e_1, e_2)$ and once for $(e, x) = (e_2, e_1)$. This yields $e_2 \circ e_1 = e_1 \circ e_2 = e_2$, but at the same time $e_1 \circ e_2 = e_2 \circ e_1 = e_1$, which is a contradiction.

The identity element equals its own inverse: Take $x = e$ in the second axiom. This yields $e \circ e = e$, which, by the last axiom, defines $e = e^{\text{inv}}$ to be its own inverse.

The inverse x^{inv} of a given element $x \in G$ is unique: Suppose $y \neq z$ are both inverses of $x \in G$. On the one hand we have $y \circ x \circ z = y \circ (x \circ z) = y \circ e = y$. But at the same time we have $y \circ x \circ z = (y \circ x) \circ z = e \circ z = z$. This is a contradiction.

The inverse of the inverse of a given element $x \in G$ reproduces that element: For ease of notation, denote the inverse of x by $y = x^{\text{inv}}$. Then $x = e \circ x = (y^{\text{inv}} \circ y) \circ x = y^{\text{inv}} \circ (y \circ x) = y^{\text{inv}} \circ e = y^{\text{inv}}$, i.e. $x = (x^{\text{inv}})^{\text{inv}}$.

(15) **b.** Consider the set of 4×4 matrices

$$G = \left\{ \begin{pmatrix} \mathrm{I} & -\Theta \\ \Theta & \mathrm{I} \end{pmatrix} \middle| \, \Theta \stackrel{\text{def}}{=} \begin{pmatrix} \theta_{11} & \theta_{12} \\ 0 & \theta_{22} \end{pmatrix} , \, \mathrm{I} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \, \theta_{ij} \in \mathbb{R} \right\} .$$

Which constraint on the parameters $\theta_{11}, \theta_{12}, \theta_{22}$ should we impose in order for G to define an abelian group under matrix multiplication?

Taking Ψ to be a matrix of the same form as Θ, but with entries ψ_{ij} instead of θ_{ij} $(i, j = 1, 2)$, we have

$$\begin{pmatrix} \mathrm{I} & -\Theta \\ \Theta & \mathrm{I} \end{pmatrix} \begin{pmatrix} \mathrm{I} & -\Psi \\ \Psi & \mathrm{I} \end{pmatrix} = \begin{pmatrix} \mathrm{I} - \Theta\Psi & -(\Theta + \Psi) \\ \Theta + \Psi & \mathrm{I} + \Theta\Psi \end{pmatrix} ,$$

thus $\Theta\Psi$ must be the 2×2 null matrix. Note that the required constraint must be formulated in terms of a condition on the generic form of Θ. Working this out for the given parametric form of Θ and Ψ yields

$$\begin{pmatrix} \theta_{11} & \theta_{12} \\ 0 & \theta_{22} \end{pmatrix} \begin{pmatrix} \psi_{11} & \psi_{12} \\ 0 & \psi_{22} \end{pmatrix} = \begin{pmatrix} \theta_{11}\psi_{11} & \theta_{11}\psi_{12} + \theta_{12}\psi_{22} \\ 0 & \theta_{22}\psi_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

This must hold for *all* coefficients, whence (take $\Theta = \Psi$ for instance) it follows that $\theta_{11} = \theta_{22} = 0$, leaving θ_{12} undetermined, whence

$$\Theta \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix} ,$$

with $\theta \in \mathbb{R}$.

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(20) 2. METRIC SPACE

Definition 3 Let V be a vector space over $\mathbb{K}$. A *distance* on V is a nondegenerate positive definite symmetric mapping $d : V \times V \longrightarrow \mathbb{R}$ such that for all $u, v, w \in V$, $\lambda \in \mathbb{K}$,

- $d(v, w) \geq 0$ and $d(v, w) = 0$ if and only if $v = w$,
- $d(v, w) = d(w, v)$,
- $d(v, w) \leq d(v, u) + d(u, w)$.

We consider (2+1)-dimensional Euclidean spacetime $M = \mathbb{R}^3 = \left\{ (\vec{x}, t) \,|\, x = (x_1, x_2) \in \mathbb{R}^2 \text{ and } t \in \mathbb{R} \right\}$, interpreted as a vector space with the usual definitions of vector addition and scalar multiplication, and furnished with a bivariate operator

$$d : M \times M \to \mathbb{R} : ((\vec{x}, t), (\vec{y}, u)) \mapsto d\left((\vec{x}, t), (\vec{y}, u)\right) \stackrel{\text{def}}{=} \begin{cases} \|\vec{y} - \vec{x}\| & \text{if } t = u, \\ |u - t| & \text{if } t \neq u, \end{cases}$$

in which $\|\vec{y} - \vec{x}\| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$, i.e. the Euclidean distance between $\vec{x}$ and $\vec{y}$. You may take it for granted that this Euclidean distance does indeed define a distance according to Definition 3.

In order to verify whether d defines a distance, please answer all problems under a, b and c below.

(10) **a.** Prove or disprove the first axiom (positivity and nondegeneracy).

(5) **b.** Prove or disprove the second axiom (symmetry).

(5) **c.** Prove or disprove the third axiom (triangle inequality).

The mapping d fails to be a distance. Symmetry and positive nondegeneracy are fulfilled:

a. $d\left((\vec{x}, t), (\vec{y}, u)\right) \geq 0$; in particular equality clearly implies $t = u$, so that $0 = d\left((\vec{x}, t), (\vec{y}, u)\right) = \|\vec{y} - \vec{x}\|$, which holds iff $\vec{x} = \vec{y}$;

b. $d\left((\vec{x}, t), (\vec{y}, u)\right) = d\left((\vec{y}, u), (\vec{x}, t)\right)$ by virtue of the symmetries $\|\vec{y} - \vec{x}\| = \|\vec{x} - \vec{y}\|$ and $|u - t| = |t - u|$.

However, the triangle inequality fails, for consider the distance between two distinct spacetime points for equal time t.

c. On the one hand we have $d\left((\vec{y}, t), (\vec{x}, t)\right) = \|\vec{y} - \vec{x}\| \stackrel{\text{def}}{=} \Delta > 0$. If $(\vec{z}, v)$ is any other spacetime point with $v \neq t$, then $d\left((\vec{x}, t), (\vec{z}, v)\right) + d\left((\vec{z}, v), (\vec{y}, t)\right) = 2|v - t|$. This may be smaller than Δ, viz. if $|v - t| < \Delta/2$.

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(**20**) **3.** DISTRIBUTION THEORY

Definition 4 The class $\mathcal{S}(\mathbb{R}^n)$ of *smooth test functions of rapid decay*, $\phi : \mathbb{R}^n \longrightarrow \mathbb{K}$, a.k.a. *Schwartz functions*, is defined as follows:

$$\phi \in \mathcal{S}(\mathbb{R}^n) \quad \text{iff} \quad \phi \in C^\infty(\mathbb{R}^n) \quad \text{and} \quad \sup_{x \in \mathbb{R}^n} |x^\alpha \nabla_\beta \phi(x)| < \infty \,,$$

for all multi-indices α and β.

Theorem 1 *If $T \in \mathcal{S}'(\mathbb{R}^n)$, then there exist a constant $c > 0$ and multi-indices α, β such that*

$$|T[\phi]| \leq c \sup_{x \in \mathbb{R}^n} |x^\alpha \nabla_\beta \phi(x)| \,.$$

(10) **a.** Prove Theorem 1 for the case of a regular tempered distribution $T \stackrel{\text{def}}{=} T_f$, i.e. a tempered distribution of the form

$$T_f[\phi] = \int_{\mathbb{R}^n} f(x)\,\phi(x)\,dx\,,$$

with $f \in L^1(\mathbb{R}^n)$.

Using the Hölder inequality (in †) we find

$$|T_f[\phi]| = |\int_{\mathbb{R}^n} f(x)\,\phi(x)\,dx| \leq \int_{\mathbb{R}^n} |f(x)\,\phi(x)|\,dx = \|f\,\phi\|_1 \stackrel{\dagger}{\leq} \|f\|_1\,\|\phi\|_\infty = c \sup_{x\in\mathbb{R}^n} |\phi(x)|\,,$$

in which $c = \|f\|_1$. Note that we may take trivial multi-indices in the estimation in this case: $\alpha = \beta = 0 \in \mathbb{N}^n$.

(10) **b.** Show that it also holds if $T \stackrel{\text{def}}{=} \nabla_\alpha T_f$, i.e. for any α-partial derivative of a tempered distribution.

The space $\mathcal{S}(\mathbb{R}^n)$ is closed under differentiation. This means that $(-1)^\beta \nabla_\beta \phi \in \mathcal{S}(\mathbb{R}^n)$ for all multi-indices β. Since, by definition, $\nabla_\beta T_f[\phi] = (-1)^\beta T_f[\nabla_\beta \phi]$ we have, using the conclusion of problem a, $|\nabla_\beta T_f[\phi]| = |T_f[\nabla_\beta \phi]| \leq c\,\sup_{x\in\mathbb{R}^n} |\nabla_\beta \phi(x)|$ for some $c \in \mathbb{R}$ and given β. Note that we may take α trivial in Theorem 1 in this case.

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(**20**) **4.** DISTRIBUTION THEORY & FOURIER ANALYSIS (EXAM APRIL 11, 2014, PROBLEM 4)

The Fourier transform $\widehat{T} \in \mathscr{S}'(\mathbb{R}^n)$ of a distribution $T \in \mathscr{S}'(\mathbb{R}^n)$ is defined as follows:

$$\widehat{T}(\phi) = T(\widehat{\phi}) \quad \text{for all test functions } \phi \in \mathscr{S}(\mathbb{R}^n), \qquad (\star)$$

in which

$$\widehat{\phi}(\omega) = \int_{\mathbb{R}^n} e^{-i\omega\cdot x}\,\phi(x)\,dx\,.$$

The purpose of this problem is to motivate this definition.

To this end, consider any function $f : \mathbb{R}^n \to \mathbb{C}$ of polynomial growth for which the Fourier integral

$$\widehat{f}(\omega) = \int_{\mathbb{R}^n} e^{-i\omega\cdot x}\,f(x)\,dx$$

is well-defined and yields a function $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ of polynomial growth. Denote by $T_f \in \mathscr{S}'(\mathbb{R}^n)$ the corresponding *regular* tempered distribution, i.e.

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\,\phi(x)\,dx$$

for all test functions $\phi \in \mathscr{S}(\mathbb{R}^n)$. It is then natural to define $\widehat{T}_f \stackrel{\text{def}}{=} T_{\widehat{f}}$, i.e.

$$\widehat{T}_f(\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \widehat{f}(\xi)\,\phi(\xi)\,d\xi\,. \qquad (*)$$

(10) **a.** Show that this definition $(*)$ implies $\widehat{T}_f(\phi) = T_f(\widehat{\phi})$, consistent with $(\star)$, for any $\phi \in \mathscr{S}(\mathbb{R}^n)$.

(*Hint:* In $(*)$ apply the Fourier reconstruction formula in the following form: $\phi(\xi) = \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} e^{i\xi\cdot x}\, \widehat{\phi}(x)\, dx$.)

Following the hint (in the quality marked by $*$) we write

$$\widehat{T}_f(\phi) \stackrel{\text{def}}{=} T_{\widehat{f}}(\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \widehat{f}(\xi)\, \phi(\xi)\, d\xi \stackrel{*}{=} \int_{\mathbb{R}^n} \widehat{f}(\xi) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot x}\, \widehat{\phi}(x)\, dx \right) d\xi\,.$$

Interchanging the order of integration this is seen to be equivalent to

$$\widehat{T}_f(\phi) = \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{i\xi\cdot x}\; d\xi \right) \widehat{\phi}(x)\, dx \stackrel{\circ}{=} \int_{\mathbb{R}^n} f(x)\, \widehat{\phi}(x)\, dx = T_f(\widehat{\phi})\,.$$

In $\circ$ the Fourier reconstruction formula has been used once more, this time for the function f.

This result justifies the general definition $(\star)$, which also holds if $T \in \mathscr{S}'(\mathbb{R}^n)$ is not regular.

As an example, consider the (non-regular) Dirac point distribution $\delta \in \mathscr{S}'(\mathbb{R}^n)$, defined by $\delta(\phi) = \phi(0)$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

(10) **b.** Use the general definition $(\star)$ to prove that $\widehat{\delta} = T_{\mathit{1}}$. Here $T_{\mathit{1}} \in \mathscr{S}'(\mathbb{R}^n)$ is the regular tempered distribution corresponding to the constant function $\mathit{1} : \mathbb{R}^n \to \mathbb{C} : x \mapsto \mathit{1}(x) = 1$.

Using the distributional definition of the Fourier transform (in $\star$) we find that

$$\widehat{\delta}(\phi) \stackrel{\star}{=} \delta(\widehat{\phi}) \stackrel{\text{def}}{=} \widehat{\phi}(0) \stackrel{\dagger}{=} \int_{\mathbb{R}^n} \phi(x)\, dx \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \mathit{1}(x)\, \phi(x)\, dx \stackrel{\text{def}}{=} T_{\mathit{1}}(\phi)\,.$$

Note that $\dagger$ uses a special case of the definition of the Fourier transform, viz.

$$\widehat{\phi}(\omega) = \int_{\mathbb{R}^n} e^{-i\omega\cdot x} \phi(x)\, dx\,,$$

for $\omega = 0 \in \mathbb{R}^n$.

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(**15**) **5.** FOURIER ANALYSIS & PDE THEORY

Consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} &= c\dfrac{\partial u}{\partial x} \\ u(x,0) &= f(x)\,, \end{cases}$$

for a function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, for a given $f \in C^1(\mathbb{R})$ and constant parameter $c \in \mathbb{R}$.

Find the solution for u in terms of f and c by Fourier transformation

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x}\, \widehat{u}(\omega,t)\, d\omega\,.$$

You may assume that f has a well-defined Fourier transform $\widehat{f}$.

Substituting the Fourier formula yields

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \left(\frac{d\widehat{u}}{dt}(\omega,t) - i\omega c\widehat{u}(\omega,t) \right) d\omega = 0\,,$$

with

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left(\widehat{u}(\omega,0)-\widehat{f}(\omega)\right)d\omega = 0\,.$$

This holds iff

$$\begin{cases} \dfrac{d\widehat{u}}{dt}(\omega,t) - i\omega c\widehat{u}(\omega,t) & = & 0\,, \\ \widehat{u}(\omega,0) & = & \widehat{f}(\omega)\,. \end{cases}$$

The solution is $\widehat{u}(\omega,t)=\widehat{f}(\omega)\,e^{i\omega ct}$. Fourier inversion yields

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(x+ct)}\,\widehat{f}(\omega)\,d\omega = f(x+ct)\,.$$

This reveals that the initial condition is a snapshot of a travelling wave with profile f propagating with velocity c (in negative x-direction if $c>0$, in positive x-direction if $c<0$, and “frozen” if $c=0$).

THE END