EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10 & 8D020. Date: Wednesday January 27 2016. Time: 13:30–16:30. Place: AUD 12.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 5 problems. The maximum credit for each item is indicated in the margin.
- The use of course notes is allowed, provided it is in immaculate state. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- Motivate your answers. You may provide your answers in Dutch or English.

GOOD LUCK!

(**25**) **1.** GROUP

Definition 1 A *group* is a collection G together with an internal operation

$$\circ : G \times G \longrightarrow G : (x, y) \mapsto x \circ y \,,$$

such that

- the operation is associative, i.e. $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in G$,
- there exists an identity element $e \in G$ such that $x \circ e = e \circ x = x$ for all $x \in G$, and
- for each $x \in G$ there exists an inverse element x^{inv} such that $x^{\text{inv}} \circ x = x \circ x^{\text{inv}} = e$.

Definition 2 Recall Definition 1. If, in addition, $x \circ y = y \circ x$ for all $x, y \in G$, then the group is called *commutative*, or *abelian*.

(10) **a.** Prove the following properties for a group G:

- The identity element $e \in G$ is unique.
- The identity element equals its own inverse: $e^{\text{inv}} = e$.
- The inverse x^{inv} of a given element $x \in G$ is unique.
- The inverse of the inverse of a given element $x \in G$ reproduces that element: $(x^{\text{inv}})^{\text{inv}} = x$.

(15) **b.** Consider the set of 4×4 matrices

$$G=\left\{\left(\begin{array}{cc} \mathrm{I} & -\Theta \\ \Theta & \mathrm{I} \end{array}\right)\middle|\,\Theta \stackrel{\mathrm{def}}{=} \left(\begin{array}{cc} \theta_{11} & \theta_{12} \\ 0 & \theta_{22} \end{array}\right)\,,\,\mathrm{I} \stackrel{\mathrm{def}}{=} \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)\,,\,\theta_{ij}\in\mathbb{R}\right\}\,.$$

Which constraint on the parameters $\theta_{11}, \theta_{12}, \theta_{22}$ should we impose in order for G to define an abelian group under matrix multiplication?

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(**20**) **2.** METRIC SPACE

Definition 3 Let V be a vector space over $\mathbb{K}$. A *distance* on V is a nondegenerate positive definite symmetric mapping $d : V \times V \longrightarrow \mathbb{R}$ such that for all $u, v, w \in V$, $\lambda \in \mathbb{K}$,

- $d(v, w) \geq 0$ and $d(v, w) = 0$ if and only if $v = w$,
- $d(v, w) = d(w, v)$,
- $d(v, w) \leq d(v, u) + d(u, w)$.

We consider (2+1)-dimensional Euclidean spacetime $M = \mathbb{R}^3 = \left\{(\vec{x}, t)\,|\, x = (x_1, x_2) \in \mathbb{R}^2 \text{ and } t \in \mathbb{R}\right\}$, interpreted as a vector space with the usual definitions of vector addition and scalar multiplication, and furnished with a bivariate operator

$$d : M \times M \to \mathbb{R} : ((\vec{x}, t), (\vec{y}, u)) \mapsto d\left((\vec{x}, t), (\vec{y}, u)\right) \stackrel{\mathrm{def}}{=} \left\{\begin{array}{ll} \|\vec{y} - \vec{x}\| & \text{if } t = u, \\ |u - t| & \text{if } t \neq u, \end{array}\right.$$

in which $\|\vec{y} - \vec{x}\| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$, i.e. the Euclidean distance between $\vec{x}$ and $\vec{y}$. You may take it for granted that this Euclidean distance does indeed define a distance according to Definition 3.

In order to verify whether d defines a distance, please answer all problems under a, b and c below.

(10) **a.** Prove or disprove the first axiom (positivity and nondegeneracy).

(5) **b.** Prove or disprove the second axiom (symmetry).

(5) **c.** Prove or disprove the third axiom (triangle inequality).

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(**20**) **3.** DISTRIBUTION THEORY

Definition 4 The class $\mathcal{S}(\mathbb{R}^n)$ of *smooth test functions of rapid decay*, $\phi : \mathbb{R}^n \longrightarrow \mathbb{K}$, a.k.a. *Schwartz functions*, is defined as follows:

$$\phi \in \mathcal{S}(\mathbb{R}^n) \quad \text{iff} \quad \phi \in C^\infty(\mathbb{R}^n) \quad \text{and} \quad \sup_{x\in\mathbb{R}^n} |x^\alpha \nabla_\beta \phi(x)| < \infty\,,$$

for all multi-indices α and β.

Theorem 1 *If $T \in \mathcal{S}'(\mathbb{R}^n)$, then there exist a constant $c > 0$ and multi-indices α, β such that*

$$|T[\phi]| \leq c \sup_{x \in \mathbb{R}^n} |x^\alpha \nabla_\beta \phi(x)| .$$

(10) **a.** Prove Theorem 1 for the case of a regular tempered distribution $T \stackrel{\text{def}}{=} T_f$, i.e. a tempered distribution of the form

$$T_f[\phi] = \int_{\mathbb{R}^n} f(x)\, \phi(x)\, dx ,$$

with $f \in L^1(\mathbb{R}^n)$.

(10) **b.** Show that it also holds if $T \stackrel{\text{def}}{=} \nabla_\alpha T_f$, i.e. for any α-partial derivative of a tempered distribution.

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(**20**) **4.** DISTRIBUTION THEORY & FOURIER ANALYSIS (EXAM APRIL 11, 2014, PROBLEM 4)

The Fourier transform $\widehat{T} \in \mathscr{S}'(\mathbb{R}^n)$ of a distribution $T \in \mathscr{S}'(\mathbb{R}^n)$ is defined as follows:

$$\widehat{T}(\phi) = T(\widehat{\phi}) \quad \text{for all test functions } \phi \in \mathscr{S}(\mathbb{R}^n), \qquad (\star)$$

in which

$$\widehat{\phi}(\omega) = \int_{\mathbb{R}^n} e^{-i\omega \cdot x}\, \phi(x)\, dx .$$

The purpose of this problem is to motivate this definition.

To this end, consider any function $f : \mathbb{R}^n \to \mathbb{C}$ of polynomial growth for which the Fourier integral

$$\widehat{f}(\omega) = \int_{\mathbb{R}^n} e^{-i\omega \cdot x}\, f(x)\, dx$$

is well-defined and yields a function $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ of polynomial growth. Denote by $T_f \in \mathscr{S}'(\mathbb{R}^n)$ the corresponding *regular* tempered distribution, i.e.

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\, \phi(x)\, dx$$

for all test functions $\phi \in \mathscr{S}(\mathbb{R}^n)$. It is then natural to define $\widehat{T}_f \stackrel{\text{def}}{=} T_{\widehat{f}}$, i.e.

$$\widehat{T}_f(\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \widehat{f}(\xi)\, \phi(\xi)\, d\xi . \qquad (*)$$

(10) **a.** Show that this definition $(*)$ implies $\widehat{T}_f(\phi) = T_f(\widehat{\phi})$, consistent with $(\star)$, for any $\phi \in \mathscr{S}(\mathbb{R}^n)$.
(*Hint:* In $(*)$ apply the Fourier reconstruction formula in the following form: $\phi(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi \cdot x}\, \widehat{\phi}(x)\, dx$.)

This result justifies the general definition $(\star)$, which also holds if $T \in \mathscr{S}'(\mathbb{R}^n)$ is not regular.

As an example, consider the (non-regular) Dirac point distribution $\delta \in \mathscr{S}'(\mathbb{R}^n)$, defined by $\delta(\phi) = \phi(0)$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

(10) **b.** Use the general definition $(\star)$ to prove that $\widehat{\delta} = T_{\mathit{1}}$. Here $T_{\mathit{1}} \in \mathscr{S}'(\mathbb{R}^n)$ is the regular tempered distribution corresponding to the constant function $\mathit{1} : \mathbb{R}^n \to \mathbb{C} : x \mapsto \mathit{1}(x) = 1$.

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(**15**) **5.** FOURIER ANALYSIS & PDE THEORY

Consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} &= c\dfrac{\partial u}{\partial x} \\ u(x,0) &= f(x) \,, \end{cases}$$

for a function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, for a given $f \in C^1(\mathbb{R})$ and constant parameter $c \in \mathbb{R}$.

Find the solution for u in terms of f and c by Fourier transformation

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x}\, \widehat{u}(\omega, t)\, d\omega \,.$$

You may assume that f has a well-defined Fourier transform $\widehat{f}$.

THE END