

## EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10. Date: Wednesday January 31, 2018. Time: 13:30–16:30. Place: PAV SH2 H.

### Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of **4 problems**. The maximum credit for each item is indicated in the margin.
- The use of course notes is allowed, provided it is in immaculate state. The use of other notes, calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- Motivate your answers. You may provide your answers in Dutch or English.

GOOD LUCK!

### (30) 1. LINEAR SPACE (EXAM JULY 8, 2004, PROBLEM 2)

$C_0^2([0, 1])$  is the class of twofold continuously differentiable, real-valued functions of type  $f : [0, 1] \rightarrow \mathbb{R}$ , for which  $f(0) = f(1) = f'(0) = f'(1) = 0$ . (By  $f'(0)$  en  $f'(1)$  we mean the right, resp. left derivative at the point of interest.) Without proof we state that  $C^2([0, 1])$ , the class of real-valued functions on the closed interval  $[0, 1]$  which are twofold continuously differentiable, constitutes a linear space.

- (5) **a.** Prove that  $C_0^2([0, 1])$  is a linear space.  
(Hint:  $C_0^2([0, 1]) \subset C^2([0, 1])$ .)

Since  $C_0^2([0, 1]) \subset C^2([0, 1])$ , with  $C^2([0, 1])$  a linear space, it suffices to prove that  $C_0^2([0, 1])$  is closed w.r.t. vector addition and scalar multiplication. Suppose  $f, g \in C_0^2([0, 1])$  and  $\lambda, \mu \in \mathbb{R}$  arbitrary, then  $\lambda f + \mu g$  is likewise twofold continuously differentiable (since, by definition,  $(\lambda f + \mu g)' = \lambda f' + \mu g'$ , etc.). In particular we have  $(\lambda f + \mu g)(r) = \lambda f(r) + \mu g(r) = 0$  and  $(\lambda f + \mu g)'(r) = \lambda f'(r) + \mu g'(r) = 0$  for boundary points  $r \in \{0, 1\}$ , so that  $\lambda f + \mu g$  also satisfies the boundary conditions, i.e.  $\lambda f + \mu g \in C_0^2([0, 1])$ .

We endow the linear space  $C_0^2([0, 1])$  with a real inner product according to one of the definitions below. The subscript refers to the applicable definition, so do not forget to indicate this in your notation throughout.

**Definition 1:** For  $f, g \in C_0^2([0, 1])$ ,

$$\langle f|g \rangle_1 = \int_0^1 f(x)g(x) dx + \int_0^1 f'(x)g'(x) dx.$$

**Definition 2:** For  $f, g \in C_0^2([0, 1])$ ,

$$\langle f|g \rangle_2 = \int_0^1 f(x)g(x) dx - \frac{1}{2} \int_0^1 f''(x)g(x) dx - \frac{1}{2} \int_0^1 f(x)g''(x) dx.$$

- (5) **b.** Show that Definition 1 is a good definition, in the sense that it indeed defines an inner product.

Suppose  $f, g, h \in C_0^2([0, 1])$  en  $\lambda, \mu \in \mathbb{R}$ . Than both  $\int_0^1 f(x)g(x) dx$  as well as  $\int_0^1 f'(x)g'(x) dx$  are well defined, so  $\langle f|g \rangle_1 \in \mathbb{R}$ . Moreover:

$$\begin{aligned}
 \langle \lambda f + \mu g|h \rangle_1 &= \int_0^1 (\lambda f + \mu g)(x) h(x) dx + \int_0^1 (\lambda f + \mu g)'(x) h'(x) dx \\
 &= \int_0^1 (\lambda f(x) + \mu g(x)) h(x) dx + \int_0^1 (\lambda f'(x) + \mu g'(x)) h'(x) dx \\
 &= \lambda \left( \int_0^1 f(x) h(x) dx + \int_0^1 f'(x) h'(x) dx \right) + \mu \left( \int_0^1 g(x) h(x) dx + \int_0^1 g'(x) h'(x) dx \right) \\
 &= \lambda \langle f|h \rangle_1 + \mu \langle g|h \rangle_1, \\
 \langle f|\lambda g + \mu h \rangle_1 &= \int_0^1 f(x) (\lambda g + \mu h)(x) dx + \int_0^1 f'(x) (\lambda g + \mu h)'(x) dx \\
 &= \int_0^1 f(x) (\lambda g(x) + \mu h(x)) dx + \int_0^1 f'(x) (\lambda g'(x) + \mu h'(x)) dx \\
 &= \lambda \left( \int_0^1 f(x) g(x) dx + \int_0^1 f'(x) g'(x) dx \right) + \mu \left( \int_0^1 f(x) h(x) dx + \int_0^1 f'(x) h'(x) dx \right) \\
 &= \lambda \langle f|g \rangle_1 + \mu \langle f|h \rangle_1, \\
 \langle f|g \rangle_1 &= \int_0^1 f(x) g(x) dx + \int_0^1 f'(x) g'(x) dx \\
 &= \langle g|f \rangle_1 \quad \text{by virtue of commutativity of ordinary multiplication,} \\
 \langle f|f \rangle_1 &= \int_0^1 (f(x))^2 dx + \int_0^1 (f'(x))^2 dx > 0 \quad \text{if } f \text{ is not the null function.}
 \end{aligned}$$

- (5) **c. Prove that both definitions are equivalent.**  
*(Hint: Partial integration.)*

Via partial integration it follows that

$$\begin{aligned}
 \int_0^1 f''(x) g(x) dx &= [f'(x) g(x)]_0^1 - \int_0^1 f'(x) g'(x) dx = - \int_0^1 f'(x) g'(x) dx \quad \text{as well as} \\
 \int_0^1 f(x) g''(x) dx &= [f(x) g'(x)]_0^1 - \int_0^1 f'(x) g'(x) dx = - \int_0^1 f'(x) g'(x) dx.
 \end{aligned}$$

The boundary terms cancel by virtue of the boundary conditions for  $f, g \in C_0^2([0, 1])$ . By substituting these identities into Definition 2 it follows that  $\langle f|g \rangle_2 = \langle f|g \rangle_1$ .

Due to equivalence you may henceforth omit subscripts:  $\langle f|g \rangle = \langle f|g \rangle_1 = \langle f|g \rangle_2$ . With the help of this inner product we introduce, for arbitrarily fixed  $h \in C_0^2([0, 1])$ , the following linear mapping  $P_h : C_0^2([0, 1]) \rightarrow C_0^2([0, 1])$ :

**Definition:**  $P_h(f) = \frac{\langle h|f \rangle}{\langle h|h \rangle} h$ .

- (5) **d. Show that  $P_h \circ P_h = P_h$ .** The infix operator  $\circ$  denotes composition.

Take any  $f \in C_0^2([0, 1])$ . Then

$$(P_h \circ P_h)(f) = P_h(P_h(f)) = \frac{\langle h|P_h(f) \rangle}{\langle h|h \rangle} h = \frac{\langle h|\frac{\langle h|f \rangle}{\langle h|h \rangle} h \rangle}{\langle h|h \rangle} h = \frac{\langle h|f \rangle}{\langle h|h \rangle} \frac{\langle h|h \rangle}{\langle h|h \rangle} h = \frac{\langle h|f \rangle}{\langle h|h \rangle} h = P_h(f).$$

In the third equality the linearity of the inner product w.r.t. its second argument has been used. The rest follows from the definition of the composition operator  $\circ$ , resp. that of  $P_h$ . Since this holds for all  $f \in C_0^2([0, 1])$  it follows that  $P_h \circ P_h = P_h$  (idempotency).

(5) e. Show that  $P_h^\dagger = P_h$ , i.e.  $\langle g|P_h f\rangle = \langle P_h g|f\rangle$  for all  $f, g \in C_0^2([0, 1])$ .

Using the bilinearity of the real inner product, the definition of  $P_h$  and a few trivial rewritings, one finds

$$\langle g|P_h f\rangle = \langle g|\frac{\langle h|f\rangle}{\langle h|h}\rangle h\rangle = \frac{\langle h|f\rangle}{\langle h|h}\langle g|h\rangle = \frac{\langle h|g\rangle}{\langle h|h}\langle h|f\rangle = \langle \frac{\langle h|g\rangle}{\langle h|h}\rangle h|f\rangle = \langle P_h g|f\rangle.$$

General properties of the real inner product have been used here in steps 2 (linearity w.r.t. second argument), 3 (symmetry) and 4 (linearity w.r.t. first argument).

Consider the following pair of functions (note that  $f(x) = f(1-x)$  and  $g(x) = g(1-x)$ ):

$$f(x) = x^4 - 2x^3 + x^2 \quad (0 \leq x \leq 1) \quad \text{and} \quad g(x) = \begin{cases} -4x^3 + 3x^2 & (0 \leq x \leq \frac{1}{2}) \\ -4(1-x)^3 + 3(1-x)^2 & (\frac{1}{2} \leq x \leq 1) \end{cases}$$

(5) f. Show that  $f, g \in C_0^2([0, 1])$ .

Polynomial functions are infinitely differentiable, so in particular  $f$  is twofold continuously differentiable. As for  $g$  we must first inspect the “suspicious” point  $x = \frac{1}{2}$  more closely:

$$\begin{aligned} \lim_{x \uparrow \frac{1}{2}} g(x) &= \frac{1}{4} \\ \lim_{x \downarrow \frac{1}{2}} g(x) &= \frac{1}{4}, \end{aligned}$$

which, as a matter of fact, also follows directly from the symmetry consideration  $g(x) = g(1-x)$ . The function  $g$  is therefore continuous (at  $x = \frac{1}{2}$  and thus everywhere). Furthermore:

$$\begin{aligned} \lim_{x \uparrow \frac{1}{2}} g'(x) &= \lim_{x \uparrow \frac{1}{2}} (-12x^2 + 6x) = 0 \\ \lim_{x \downarrow \frac{1}{2}} g'(x) &= \lim_{x \downarrow \frac{1}{2}} (12(1-x)^2 - 6(1-x)) = 0. \end{aligned}$$

The function  $g$  is therefore continuously differentiable at  $x = \frac{1}{2}$ , with  $g'(\frac{1}{2}) = 0$ . Moreover:

$$\begin{aligned} \lim_{x \uparrow \frac{1}{2}} g''(x) &= \lim_{x \uparrow \frac{1}{2}} (-24x + 6) = -6 \\ \lim_{x \downarrow \frac{1}{2}} g''(x) &= \lim_{x \downarrow \frac{1}{2}} (-24(1-x) + 6) = -6. \end{aligned}$$

The function  $g'$  is therefore also continuously differentiable at  $x = \frac{1}{2}$ , with  $g''(\frac{1}{2}) = -6$ . All in all it follows that  $g$  is twofold continuously differentiable at  $x = \frac{1}{2}$  and thus everywhere. Finally we need to check the boundary conditions: We have  $f'(x) = 4x^3 - 6x^2 + 2x$  and  $g'(x) = -12x^2 + 6x$  for  $x < \frac{1}{2}$  and  $g'(x) = -g'(1-x)$  for  $x > \frac{1}{2}$ , so  $f(0) = f(1) = f'(0) = f'(1) = 0$  and likewise for  $g$ , in which the boundary derivatives have been defined as follows:

$$\begin{aligned} f'(0) &\stackrel{\text{def}}{=} \lim_{x \downarrow 0} f'(x) \\ f'(1) &\stackrel{\text{def}}{=} \lim_{x \uparrow 1} f'(x). \end{aligned}$$



## (25) 2. NORM

Consider the  $p$ -parametrized family of norms on the vector space  $\mathbb{R}^2$ ,

$$\|\cdot\|_p : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto \|(x, y)\|_p \stackrel{\text{def}}{=} (|x|^p + |y|^p)^{1/p},$$

for  $p \geq 1$ , together with the formal limit

$$\| \cdot \|_{\infty} : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto \| (x, y) \|_{\infty} \stackrel{\text{def}}{=} \max(|x|, |y|).$$

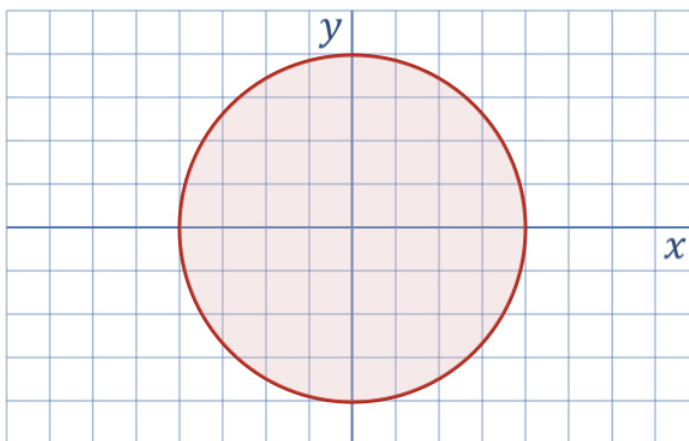
- (5) **a. Prove:**  $\lim_{p \rightarrow \infty} (|x|^p + |y|^p)^{1/p} = \max(|x|, |y|)$ .

We have the following inclusion:

$$\max(|x|, |y|) \leq (|x|^p + |y|^p)^{1/p} = \max(|x|, |y|) \left( \frac{|x|^p + |y|^p}{\max(|x|, |y|)^p} \right)^{1/p} \leq 2^{1/p} \max(|x|, |y|).$$

Taking the limit  $p \rightarrow \infty$  (noting that  $\lim_{p \rightarrow \infty} 2^{1/p} = 1$ ) yields the result by virtue of the inclusion theorem.

For each  $p \geq 1$ , including the formal limit  $p = \infty$ , the *unit circle*  $C_p$  and the (open) *unit disk*  $D_p$  are defined as the sets  $C_p : \|(x, y)\|_p = 1$ , respectively  $D_p : \|(x, y)\|_p < 1$ , with  $(x, y) \in \mathbb{R}^2$ . The figure below illustrates the cases  $C_2$  and  $D_2$ .



GRAPH OF  $C_2 : \sqrt{x^2 + y^2} = 1$  AND ITS INTERIOR  $D_2 : \sqrt{x^2 + y^2} < 1$ .

- (10) **b. Sketch in the same figure (cf. appendix) the graph of the unit circles  $C_1 : \|(x, y)\|_1 = 1$  and  $C_{\infty} : \|(x, y)\|_{\infty} = 1$ , and clearly indicate which one is which.**

For the case  $p = 1$  we have the equation  $\|(x, y)\|_1 = 1$ , i.e.  $|x| + |y| = 1$ . Depending on the quadrant  $(x, y)$  lies in this is the equation of a line of the type  $\pm x \pm y = 1$ , with slope  $\pm 1$ , which, when constrained to that particular quadrant, defines a finite line segment with end points on the  $x$ - and  $y$ -axes. Thus  $C_1$  is a square. For the case  $p = \infty$  we have the equation  $\|(x, y)\|_{\infty} = 1$ , i.e.  $\max(|x|, |y|) = 1$ . Drawing the graphs of the extremal cases  $|x| = 1$  (two vertical lines,  $x = \pm 1$ ) and  $|y| = 1$  (two horizontal lines  $y = \pm 1$ ), we obtain the set  $S = \{(x, y) \in \mathbb{R}^2 \mid |x| = 1 \wedge |y| = 1\}$ . Confined to the region  $R = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1 \wedge |y| \leq 1\}$  we find  $C_{\infty} = S \cap R$  as the intersection. Thus  $C_{\infty}$  is likewise a square.

The unit circle  $C_p$  is called *convex* if  $(x, y), (u, v) \in C_p$  implies  $\lambda(x, y) + (1 - \lambda)(u, v) \in \overline{D_p} = D_p \cup C_p$  for all  $\lambda \in [0, 1]$ .

- (5) **c. What does convexity of  $C_p$  mean graphically?**  
*(Hint: Consider the line piece connecting two endpoints  $(x, y), (u, v) \in C_p$ .)*

The line piece  $\ell$  suggested in the hint can be parametrized as  $\ell : \lambda(x, y) + (1 - \lambda)(u, v) = 1$ , with restricted parameter domain  $\lambda \in [0, 1]$ , i.e.  $\ell$  linearly interpolates  $(x, y)$  and  $(u, v)$ . The definition of convexity states that  $\ell$  lies entirely inside the closure  $\overline{D}_p$  of the interior of  $C_p$ , cf. the appendix for an illustration.

(5) **d.** Show that  $C_p$  is convex for any  $p \geq 1$  including  $p = \infty$ .

(Hint: You may use the fact that  $\| \cdot \|_p$  for  $p \geq 1$  and  $\| \cdot \|_\infty$  define norms without further proof.)

To prove the statement in c, consider  $(x, y), (u, v) \in C_p$ , so that  $\|(x, y)\|_p = \|(u, v)\|_p = 1$ . Then, for  $\lambda \in [0, 1]$ ,

$$\|\lambda(x, y) + (1 - \lambda)(u, v)\|_p \leq \|\lambda(x, y)\|_p + \|(1 - \lambda)(u, v)\|_p = |\lambda|\|(x, y)\|_p + |1 - \lambda|\|(u, v)\|_p = |\lambda| + |1 - \lambda| \leq 1,$$

in which we have used the triangle inequality for a norm (first inequality) and the absolute homogeneity axiom of a norm (first equality). This shows that, indeed,  $\lambda(x, y) + (1 - \lambda)(u, v) \in \overline{D}_p$ , so that, by definition  $C_p$  is convex.



## (20) 3. MAX-PLUS ALGEBRA

Consider the set  $\mathbb{A} = \mathbb{R} \cup \{(-\infty)\}$  consisting of all real numbers and the formal element ‘ $(-\infty)$ ’. Please adhere to the use of parentheses to avoid confusion.

Commutative addition  $\oplus : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$  is defined in terms of the maximum operator as follows:

$$a \oplus b = \max(a, b) \quad \text{for all } a, b \in \mathbb{A},$$

with the ‘natural’ definition  $\max(a, (-\infty)) = \max((-\infty), a) = a$  for any  $a \in \mathbb{A}$ .

Commutative multiplication  $\otimes : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$  is defined as follows:

$$a \otimes b = a + b \quad \text{for all } a \in \mathbb{A},$$

with the ‘natural’ definition  $(-\infty) + a = a + (-\infty) = (-\infty)$  for any  $a \in \mathbb{A}$ .

(5) **a.** Show that  $\oplus$  is idempotent, i.e. show that  $a \oplus a = a$  for all  $a \in \mathbb{A}$ .

Clearly  $a \oplus a \stackrel{\text{def}}{=} \max(a, a) = a$  if  $a \in \mathbb{R}$ . But, by definition (viz.  $\max(a, (-\infty)) = \max((-\infty), a) = a$  for any  $a \in \mathbb{A}$ ), this equality also holds for  $a = (-\infty)$ .

(2 $\frac{1}{2}$ ) **b1.** Show that  $\oplus$  is associative on  $\mathbb{A}$ .

For  $a, b, c \in \mathbb{R}$  we have, by associativity of the max-operator,

$$(a \oplus b) \oplus c \stackrel{\text{def}}{=} \max(a, b) \oplus c \stackrel{\text{def}}{=} \max(\max(a, b), c) = \max(a, \max(b, c)) \stackrel{\text{def}}{=} a \oplus \max(b, c) \stackrel{\text{def}}{=} a \oplus (b \oplus c).$$

The case  $a = (-\infty)$  requires verification of correctness of each of the steps in the above sequence. This follows immediately with the help of the defining property  $\max(a, (-\infty)) = \max((-\infty), a) = a$  for any  $a \in \mathbb{A}$ , viz.

$$((-\infty) \oplus b) \oplus c \stackrel{\text{def}}{=} \max((-\infty), b) \oplus c \stackrel{\text{def}}{=} \max(b, c) \stackrel{\text{def}}{=} \max((-\infty), \max(b, c)) \stackrel{\text{def}}{=} (-\infty) \oplus \max(b, c) \stackrel{\text{def}}{=} (-\infty) \oplus (b \oplus c).$$

The cases  $b = (-\infty)$  and  $c = (-\infty)$  are analogous. In fact, by commutativity of  $\oplus$ , these two cases are implied by the one proven above.

(2 $\frac{1}{2}$ ) **b2.** Show that  $\otimes$  is associative on  $\mathbb{A}$ .

We have, for all  $a, b, c \in \mathbb{R}$ , by associativity of ordinary addition,

$$(a \otimes b) \otimes c \stackrel{\text{def}}{=} (a + b) + c = a + (b + c) \stackrel{\text{def}}{=} a \otimes (b \otimes c).$$

Again, we need to verify this identity in the nontrivial case in which any of the  $a, b$  or  $c$  correspond(s) to the infinite element  $(-\infty)$ . Suppose  $a = (-\infty)$ , then, using  $(-\infty) \otimes a \stackrel{\text{def}}{=} a$  for all  $a \in \mathbb{A}$ , we have

$$((-\infty) \otimes b) \otimes c \stackrel{\text{def}}{=} b \otimes c \stackrel{\text{def}}{=} (-\infty) \otimes (b \otimes c).$$

Again, the other cases,  $b = (-\infty)$  or  $c = (-\infty)$  are implied by this, exploiting commutativity of  $\otimes$ .

(5) **c. Prove the distributivity rule  $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$  for all  $a, b, c \in \mathbb{A}$ .**

We have

$$(a \oplus b) \otimes c \stackrel{\text{def}}{=} \max(a, b) + c = \max(a + c, b + c) \stackrel{\text{def}}{=} (a \otimes c) \oplus (b \otimes c).$$

(2 $\frac{1}{2}$ ) **d1. What is the null element for  $\oplus$  ?**

The null element is  $(-\infty) \in \mathbb{A}$ , since by definition  $(-\infty) \oplus a = a \oplus (-\infty) = a$  for all  $a \in \mathbb{A}$ .

(2 $\frac{1}{2}$ ) **d2. What is the unit element for  $\otimes$  ?**

The unit element is  $0 \in \mathbb{R} \subset \mathbb{A}$ , since  $0 \otimes a = a \otimes 0 = a$  for all  $a \in \mathbb{A}$  by virtue of the fact that  $\otimes \equiv +$  if  $a \in \mathbb{R}$ . That this also holds if  $a = (-\infty)$  follows immediately by setting  $b = 0$  in the definition  $(-\infty) \oplus b = b \oplus (-\infty) = b$ .



## (25) 4. DISTRIBUTION THEORY AND FOURIER ANALYSIS

Consider the ordinary differential equation (ODE)

$$u' + u = \delta,$$

in which  $u \in \mathcal{S}'(\mathbb{R})$  is assumed to be a tempered distribution. We denote the Fourier transform of  $u$  by  $\hat{u} \in \mathcal{S}'(\mathbb{R})$ . Corresponding ‘functions under the integral’ (including formal functions of Dirac type) are referred to by the same name, i.e.  $u : \mathbb{R} \rightarrow \mathbb{C}$ , respectively  $\hat{u} : \mathbb{R} \rightarrow \mathbb{C}$ .

(5) **a1. Use Fourier theory to reformulate the ODE for  $u$  into an algebraic equation for  $\hat{u}$ .**

If  $u' + u - \delta = 0$  then  $\mathcal{F}(u' + u - \delta) = \mathcal{F}(u') + \mathcal{F}(u) - \mathcal{F}(\delta) = 0$ , i.e.  $(i\omega + 1)\hat{u}(\omega) = 1$ . Here  $\mathcal{F}$  denotes the (linear) Fourier transformation operator, and  $\hat{u}$  is an abbreviation of  $\mathcal{F}(u)$ . Here we have made use of the identities  $\mathcal{F}(u')(\omega) = i\omega\mathcal{F}(u)(\omega)$  and  $\mathcal{F}(\delta)(\omega) = 1$  for all  $\omega \in \mathbb{R}$  (and appropriate Fourier convention).

(5) **a2. Show that  $\text{Re } \hat{u}(\omega) = \frac{1}{1 + \omega^2}$  and  $\text{Im } \hat{u}(\omega) = -\frac{\omega}{1 + \omega^2}$  by solving this equation for  $\hat{u} \in \mathcal{S}'(\mathbb{R})$  in Fourier space.**

With  $u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}(\omega) d\omega$  we find  $(i\omega + 1)\hat{u}(\omega) = 1$ . Solution:  $\hat{u}(\omega) = \frac{1}{i\omega + 1} = \frac{1 - i\omega}{1 + \omega^2} = \frac{1}{1 + \omega^2} - i \frac{\omega}{1 + \omega^2}$ .

Suppose  $u \in \mathcal{S}'(\mathbb{R})$  is a solution to the ODE corresponding to a regular tempered distribution.

**b.** Show that the corresponding ‘function under the integral’  $u : \mathbb{R} \rightarrow \mathbb{C} : x \mapsto u(x)$  must satisfy

$$(5) \quad \mathbf{b1.} \quad \int_{-\infty}^{\infty} u(x) dx = 1.$$

We have  $\int_{-\infty}^{\infty} u(x) dx = \hat{u}(0) \stackrel{\text{a}}{=} 1$ .

$$(5) \quad \mathbf{b2.} \quad u(0) = \frac{1}{2}.$$

(Hint: Use a2.)

We have  $u(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+i\omega} d\omega \stackrel{*}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega = \frac{1}{2\pi} \arctan \omega \Big|_{-\infty}^{\infty} = \frac{1}{2\pi} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{1}{2}$ . The step marked by \* makes use of a2 and of antisymmetry of  $\text{Im } \hat{u}(\omega)$ .

The function  $\theta : \mathbb{R} \rightarrow \mathbb{C} : x \mapsto \theta(x)$  is given by  $\theta(x) = 0$  if  $x < 0$ ,  $\theta(0) = \frac{1}{2}$ ,  $\theta(x) = 1$  if  $x > 0$ .

$$(5) \quad \mathbf{c.} \quad \text{Show that } u(x) = \theta(x) e^{-x} \text{ is a solution to the ODE.}$$

One way to see this is to Fourier transform  $u$ :  $\hat{u}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} u(x) dx = \int_0^{\infty} e^{-(1+i\omega)x} dx = -\frac{e^{-(1+i\omega)x}}{1+i\omega} \Big|_0^{\infty} = \frac{1}{1+i\omega}$ . This is indeed the function obtained under a. A somewhat more cumbersome way to arrive at the same result is to Fourier invert the solution obtained under a, which requires the Residue Theorem from complex function theory. In order to evaluate

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{1+i\omega} d\omega,$$

note that the integrand, upon extension into the  $\mathbb{C}$ -plane, has a pole at  $z = i$  in the upper half  $\mathbb{C}$ -plane. For  $x < 0$  the integration contour can be closed over an infinite half-circle via the lower half  $\mathbb{C}$ -plane:

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{1+i\omega} d\omega = \frac{1}{2\pi} \oint \frac{e^{izx}}{1+iz} dz = 0.$$

For  $x > 0$  the integration contour can be closed over an infinite half-circle via the upper half  $\mathbb{C}$ -plane:

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{1+i\omega} d\omega = \frac{1}{2\pi} \oint \frac{e^{izx}}{1+iz} dz = \frac{1}{2\pi} \left( 2\pi i \lim_{z \rightarrow i} (z-i) \frac{e^{izx}}{1+iz} \right) = e^{-x}.$$

The value at  $x=0$  has been obtained under b2.

The above proof is ‘semi-classical’, employing ‘naked functions’. A better and rather straightforward proof relies entirely on distribution theory, and goes as follows. Identify  $u \sim T_u \in \mathcal{S}'(\mathbb{R})$  as usual, and interpret the ODE as  $T'_u + T_u = \delta$ . To prove that the function-under-the-integral  $u(x) = \theta(x) e^{-x}$  is indeed the correct one defining the tempered distributional solution  $T_u$ , insert an arbitrary test function  $\phi \in \mathcal{S}(\mathbb{R})$ , and verify the ODE for this  $\phi$ :

$$T'_u(\phi) \stackrel{\text{def}}{=} -T_u(\phi') \stackrel{\text{def}}{=} -\int_{-\infty}^{\infty} u(x) \phi'(x) dx \stackrel{\text{def}}{=} -\int_0^{\infty} e^{-x} \phi'(x) dx = -e^{-x} \phi(x) \Big|_0^{\infty} - \int_0^{\infty} e^{-x} \phi(x) dx = \phi(0) - \int_{-\infty}^{\infty} \theta(x) e^{-x} \phi(x) dx.$$

The second last step follows by partial integration. One recognizes the r.h.s. as

$$\phi(0) - \int_{-\infty}^{\infty} \theta(x) e^{-x} \phi(x) dx \stackrel{\text{def}}{=} \delta(\phi) - T_u(\phi) \stackrel{\text{def}}{=} (\delta - T_u)(\phi).$$

Since this holds for all  $\phi \in \mathcal{S}(\mathbb{R})$ , this implies  $T'_u = \delta - T_u$ , which completes the proof.

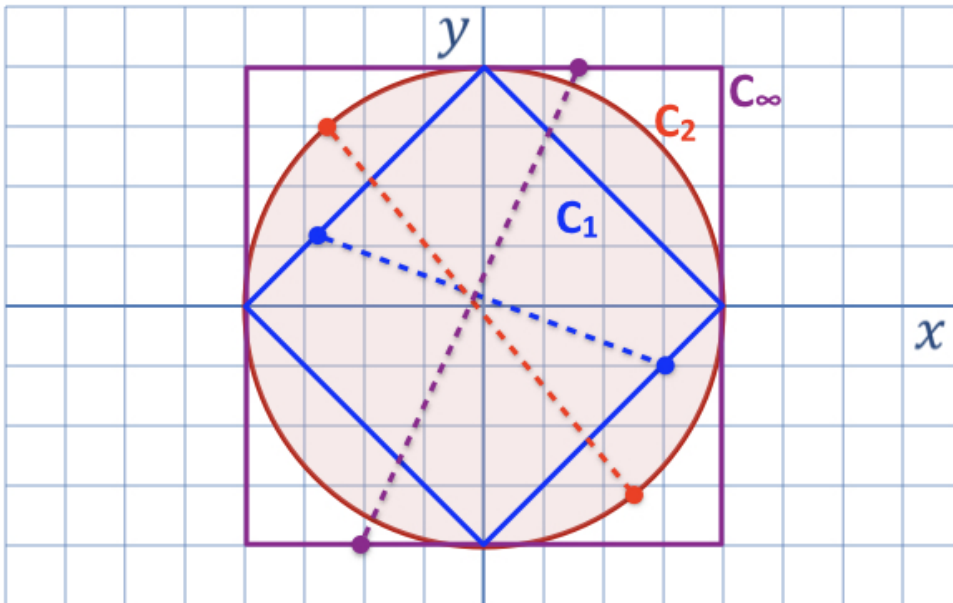


**APPENDIX**

Course code: 2DMM10. Date: Wednesday January 31, 2018. Time: 13:30–16:30. Place: PAV SH2 H.

- Name: .....
- Student ID: .....

**Write your name and student ID on this appendix and hand it in together with the rest of your answers.**



GRAPHS OF  $C_1 : |x| + |y| = 1$ ,  $C_2 : \sqrt{x^2 + y^2} = 1$  (AND INTERIOR), AND  $C_\infty : \max(|x|, |y|) = 1$ .

IN CORRESPONDING COLORS:  $\ell_p : \lambda(x, y) + (1-\lambda)(u, v) = 1$ , WITH  $\lambda \in [0, 1]$  AND FIXED  $(x, y), (u, v) \in C_p$ .