EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 2DMM10. Date: Wednesday January 31, 2018. Time: 13:30–16:30. Place: PAV SH2 H.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- The use of course notes is allowed, provided it is in immaculate state. The use of other notes, calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- Motivate your answers. You may provide your answers in Dutch or English.

GOOD LUCK!

(**30**) **1.** LINEAR SPACE (EXAM JULY 8, 2004, PROBLEM 2)

$C_0^2([0,1])$ is the class of twofold continuously differentiable, real-valued functions of type $f : [0,1] \to \mathbb{R}$, for which $f(0) = f(1) = f'(0) = f'(1) = 0$. (By $f'(0)$ en $f'(1)$ we mean the right, resp. left derivative at the point of interest.) Without proof we state that $C^2([0,1])$, the class of real-valued functions on the closed interval $[0,1]$ which are twofold continuously differentiable, constitutes a linear space.

(5) **a.** Prove that $C_0^2([0,1])$ is a linear space.
(*Hint:* $C_0^2([0,1]) \subset C^2([0,1])$.)

We endow the linear space $C_0^2([0,1])$ with a real inner product according to one of the definitions below. The subscript refers to the applicable definition, so do not forget to indicate this in your notation throughout.

Definition 1: For $f, g \in C_0^2([0,1])$,

$$\langle f|g\rangle_1 = \int_0^1 f(x)\, g(x)\, dx + \int_0^1 f'(x)\, g'(x)\, dx\,.$$

Definition 2: For $f, g \in C_0^2([0,1])$,

$$\langle f|g\rangle_2 = \int_0^1 f(x)\, g(x)\, dx - \frac{1}{2}\int_0^1 f''(x)\, g(x)\, dx - \frac{1}{2}\int_0^1 f(x)\, g''(x)\, dx\,.$$

(5) **b.** Show that Definition 1 is a good definition, in the sense that it indeed defines an inner product.

(5) **c.** Prove that both definitions are equivalent.
(*Hint:* Partial integration.)

Due to equivalence you may henceforth omit subscripts: $\langle f|g\rangle = \langle f|g\rangle_1 = \langle f|g\rangle_2$. With the help of this inner product we introduce, for arbitrarily fixed $h \in C_0^2([0,1])$, the following linear mapping $P_h : C_0^2([0,1]) \to C_0^2([0,1])$:

Definition: $P_h(f) = \dfrac{\langle h|f\rangle}{\langle h|h\rangle}\, h$.

(5) **d.** Show that $P_h \circ P_h = P_h$. The infix operator $\circ$ denotes composition.

(5) **e.** Show that $P_h^\dagger = P_h$, i.e. $\langle g|P_h f\rangle = \langle P_h g|f\rangle$ for all $f, g \in C_0^2([0,1])$.

Consider the following pair of functions (note that $f(x) = f(1-x)$ and $g(x) = g(1-x)$):

$$f(x) = x^4 - 2x^3 + x^2 \quad (0 \le x \le 1) \quad \text{and} \quad g(x) = \begin{cases} -4x^3 + 3x^2 & (0 \le x \le \frac{1}{2}) \\ -4(1-x)^3 + 3(1-x)^2 & (\frac{1}{2} \le x \le 1) \end{cases}$$

(5) **f.** Show that $f, g \in C_0^2([0,1])$.

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(25) 2. NORM

Consider the p-parametrized family of norms on the vector space $\mathbb{R}^2$,

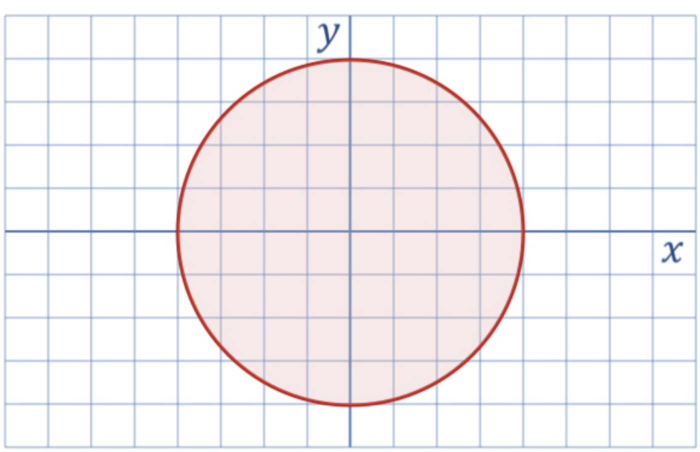
$$\|\,.\,\|_p : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto \|\,(x,y)\,\|_p \stackrel{\text{def}}{=} (|x|^p + |y|^p)^{1/p}\,,$$

for $p \ge 1$, together with the formal limit

$$\|\,.\,\|_\infty : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto \|\,(x,y)\,\|_\infty \stackrel{\text{def}}{=} \max(|x|,|y|)\,.$$

(5) **a.** Prove: $\lim\limits_{p\to\infty} (|x|^p + |y|^p)^{1/p} = \max(|x|,|y|)$.

For each $p \ge 1$, including the formal limit $p = \infty$, the *unit circle* C_p and the (open) *unit disk* D_p are defined as the sets $C_p : \|(x,y)\|_p = 1$, respectively $D_p : \|(x,y)\|_p < 1$, with $(x,y) \in \mathbb{R}^2$. The figure below illustrates the cases C_2 and D_2.



GRAPH OF $C_2 : \sqrt{x^2+y^2} = 1$ AND ITS INTERIOR $D_2 : \sqrt{x^2+y^2} < 1$.

(10) **b.** Sketch in the same figure (cf. appendix) the graph of the unit circles $C_1 : \|(x,y)\|_1 = 1$ and $C_\infty : \|(x,y)\|_\infty = 1$, and clearly indicate which one is which.

The unit circle C_p is called *convex* if $(x,y),(u,v) \in C_p$ implies $\lambda(x,y)+(1-\lambda)(u,v) \in \overline{D}_p = D_p \cup C_p$ for all $\lambda \in [0,1]$.

(5) **c.** What does convexity of C_p mean *graphically*?
(*Hint:* Consider the line piece connecting two endpoints $(x,y),(u,v) \in C_p$.)

(5) **d.** Show that C_p is convex for any $p \geq 1$ including $p = \infty$.
(*Hint:* You may use the fact that $\| \,.\, \|_p$ for $p \geq 1$ and $\| \,.\, \|_\infty$ define norms without further proof.)

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(**20**) **3.** MAX-PLUS ALGEBRA

Consider the set $\mathbb{A} = \mathbb{R} \cup \{(-\infty)\}$ consisting of all real numbers and the formal element '$(-\infty)$'. Please adhere to the use of parentheses to avoid confusion.

Commutative addition $\oplus : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is defined in terms of the maximum operator as follows:

$$a \oplus b = \max(a,b) \quad \text{for all } a,b \in \mathbb{A},$$

with the 'natural' definition $\max(a,(-\infty)) = \max((-\infty),a) = a$ for any $a \in \mathbb{A}$.

Commutative multiplication $\otimes : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is defined as follows:

$$a \otimes b = a + b \quad \text{for all } a \in \mathbb{A},$$

with the 'natural' definition $(-\infty) + a = a + (-\infty) = (-\infty)$ for any $a \in \mathbb{A}$.

(5) **a.** Show that $\oplus$ is idempotent, i.e. show that $a \oplus a = a$ for all $a \in \mathbb{A}$.

($2\frac{1}{2}$) **b1.** Show that $\oplus$ is associative on $\mathbb{A}$.

($2\frac{1}{2}$) **b2.** Show that $\otimes$ is associative on $\mathbb{A}$.

(5) **c.** Prove the distributivity rule $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$ for all $a,b,c \in \mathbb{A}$.

($2\frac{1}{2}$) **d1.** What is the null element for $\oplus$?

($2\frac{1}{2}$) **d2.** What is the unit element for $\otimes$?

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(**25**) **4.** DISTRIBUTION THEORY AND FOURIER ANALYSIS

Consider the ordinary differential equation (ODE)

$$u' + u = \delta\,,$$

in which $u \in \mathscr{S}'(\mathbb{R})$ is assumed to be a tempered distribution. We denote the Fourier transform of u by $\hat{u} \in \mathscr{S}'(\mathbb{R})$. Corresponding 'functions under the integral' (including formal functions of Dirac type) are referred to by the same name, i.e. $u : \mathbb{R} \to \mathbb{C}$, respectively $\hat{u} : \mathbb{R} \to \mathbb{C}$.

(5) **a1.** Use Fourier theory to reformulate the ODE for u into an algebraic equation for $\hat{u}$.

(5) **a2.** Show that $\operatorname{Re}\hat{u}(\omega) = \dfrac{1}{1+\omega^2}$ and $\operatorname{Im}\hat{u}(\omega) = -\dfrac{\omega}{1+\omega^2}$ by solving this equation for $\hat{u} \in \mathscr{S}'(\mathbb{R})$ in Fourier space.

Suppose $u \in \mathscr{S}'(\mathbb{R})$ is a solution to the ODE corresponding to a regular tempered distribution.

b. Show that the corresponding 'function under the integral' $u : \mathbb{R} \to \mathbb{C} : x \mapsto u(x)$ must satisfy

(5) **b1.** $\displaystyle\int_{-\infty}^{\infty} u(x)\,dx = 1.$

(5) **b2.** $u(0) = \dfrac{1}{2}$.
(*Hint:* Use a2.)

The function $\theta : \mathbb{R} \to \mathbb{C} : x \mapsto \theta(x)$ is given by $\theta(x) = 0$ if $x<0$, $\theta(0) = \dfrac{1}{2}$, $\theta(x) = 1$ if $x>0$.

(5) **c.** Show that $u(x) = \theta(x)\,e^{-x}$ is a solution to the ODE.

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APPENDIX

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- Name: ....
- Student ID: ....

Write your name and student ID on this appendix and hand it in together with the rest of your answers.