EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2F800. Date: Friday July 1, 2011. Time: 14h00–17h00. Place: HG 10.01 C 3

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- You may consult the course notes "Tensorrekening en Differentiaalmeetkunde (2F800)" by Jan de Graaf, and the draft notes "Tensor Calculus and Differential Geometry (2F800)" by Luc Florack without warranty.

(**30**) **1.** PULL BACK OF INNER PRODUCT.

We consider two inner product spaces, E and F, and a linear mapping $A : E \longrightarrow F$. The inner product on F is denoted by $G : F \times F \longrightarrow \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto G(\mathbf{x}, \mathbf{y})$, and its *pull back under* A is defined as $A^*G : E \times E \longrightarrow \mathbb{R} : (\mathbf{u}, \mathbf{v}) \mapsto A^*G(\mathbf{u}, \mathbf{v}) = G(A(\mathbf{u}), A(\mathbf{v}))$.

(10) **a.** Demonstrate with the help of an example that if we impose no further conditions on the linear mapping A, then A^*G is not necessarily an inner product on E. (*Hint:* Consider $E = \mathbb{R}^3$, $F = \mathbb{R}^2$.)

Take $E = \mathbb{R}^3$, $F = \mathbb{R}^2$, with Euclidean metric G, and $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 : (v^1, v^2, v^3) \mapsto (v^1, v^2)$ relative to a standard Cartesian basis. Then $A^*G : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R} : (\mathbf{u}, \mathbf{v}) \mapsto u^1v^1 + u^2v^2$. This is not an inner product as it fails to be non-degenerate. Note that A is surjective, but not one-to-one.

(10) **b.** Show that under suitable condition on A, A^*G is an inner product on E, and state this condition.

Symmetry: $A^*G(\mathbf{u}, \mathbf{v}) = G(A(\mathbf{u}), A(\mathbf{v})) = G(A(\mathbf{v}), A(\mathbf{u})) = A^*G(\mathbf{v}, \mathbf{u})$ for all $\mathbf{u}, \mathbf{v} \in E$.

Linearity: $A^*G(\lambda\mathbf{u} + \mu\mathbf{v}, \mathbf{w}) = G(A(\lambda\mathbf{u} + \mu\mathbf{v}), A(\mathbf{w})) = G(\lambda A(\mathbf{u}) + \mu A(\mathbf{v}), A(\mathbf{w})) = \lambda G(A(\mathbf{u}), A(\mathbf{w})) + \mu G(A(\mathbf{v}), A(\mathbf{w})) = \lambda A^*G(\mathbf{u}, \mathbf{w}) + \mu A^*G(\mathbf{v}, \mathbf{w})$.

Non-negativity: $A^*G(\mathbf{v}, \mathbf{v}) = G(A(\mathbf{v}), A(\mathbf{v})) \geq 0$. Non-degeneracy: $A^*G(\mathbf{v}, \mathbf{v}) = G(A(\mathbf{v}), A(\mathbf{v})) = 0$ iff $A(\mathbf{v}) = 0$. If A is one-to-one this is equivalent to $\mathbf{v} = 0$, and hence in this case A^*G defines an inner product.

Let A^i_j be the mixed holor of A, i.e. if $\mathbf{v} = v^i\mathbf{e}_i$ relative to basis $\{\mathbf{e}_i\}$ of E, then $A(\mathbf{v}) = v^iA^j_i\mathbf{f}_j$ relative to basis $\{\mathbf{f}_i\}$ of F. Furthermore we define the covariant holors of the metric tensors, $g_{ij} = G(\mathbf{f}_i, \mathbf{f}_j)$, and $h_{ij} = A^*G(\mathbf{e}_i, \mathbf{e}_j)$.

(10) **c.** Show that $h_{ij} = A^k_iA^\ell_jg_{k\ell}$.

Using $A(\mathbf{e}_i) = A^k_i\mathbf{f}_k$ we obtain $h_{ij} = A^*G(\mathbf{e}_i, \mathbf{e}_j) = G(A(\mathbf{e}_i), A(\mathbf{e}_j)) = G(A^k_i\mathbf{f}_k, A^\ell_j\mathbf{f}_\ell) = A^k_iA^\ell_jG(\mathbf{f}_k, \mathbf{f}_\ell) = A^k_iA^\ell_jg_{k\ell}$.

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(**40**) **2.** DEFORMATION AND STRAIN TENSORS.

We consider a smoothly deformable object in n-dimensional Euclidean space (which may be interpreted as a vector space with arbitrary origin, isomorphic to any of its local tangent spaces). The collection of material points labelled by their initial positions at time t_0 constitutes the *reference configuration*.

We consider space furnished with a coordinate system, denoting the reference configuration by $\mathrm{M} \subset \mathbb{R}^n$. Each material point is thus labeled by a coordinate tuple $\mathbf{X} \in \mathrm{M}$.

At (fixed) time $t > t_0$ the object has undergone a deformation due to internal and external forces. The details of the forces are irrelevant in this problem. The deformation causes a smooth displacement of material points, which can be described by the following *deformation map*[1]:

$$f : \mathrm{M} \longrightarrow \mathrm{N} : \mathbf{X} \mapsto \mathbf{x} = \mathbf{x}(\mathbf{X}, t)\,,$$

with $\mathbf{X} = \mathbf{x}(\mathbf{X}, t_0)$. Here $\mathrm{N} \subset \mathbb{R}^n$ denotes the new configuration, and $\mathbf{x} \in \mathrm{N}$ the new position of the material point with reference label $\mathbf{X} \in \mathrm{M}$.

Although M and N are both embedded in the same n-dimensional Euclidean space, it is convenient to use independent coordinatizations for each. In particular we thus assume no *a priori* relationship between the local bases $\{\mathbf{e}_i\}$ of $\mathrm{TM}_{\mathbf{X}}$ and $\{\mathbf{f}_\alpha\}$ of $\mathrm{TN}_{\mathbf{x}}$.

On M the components of the metric tensor relative to the reference basis $\{\mathbf{e}_i\}$ are $g_{ij} = (\mathbf{e}_i|\mathbf{e}_j)_{\mathrm{TM}_{\mathbf{X}}}$, with dual g^{ij}. On N the components of the metric tensor relative to the new basis $\{\mathbf{f}_\alpha\}$ are $h_{\alpha\beta} = (\mathbf{f}_\alpha|\mathbf{f}_\beta)_{\mathrm{TN}_{\mathbf{x}}}$, with dual $h^{\alpha\beta}$.

The *deformation tensor* is the differential $f_* \stackrel{\text{def}}{=} df$ of the deformation map. Its components relative to $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_\alpha\}$ are indicated by $F^\alpha_i = \dfrac{\partial x^\alpha}{\partial X^i}$, i.e.

$$f_* : \mathrm{TM}_{\mathbf{X}} \longrightarrow \mathrm{TN}_{\mathbf{x}} : \mathbf{v} \mapsto \mathbf{w} = f_*(\mathbf{v}) = df(\mathbf{v}) = v^i F^\alpha_i \mathbf{f}_\alpha\,,$$

with $\mathbf{v} = v^i \mathbf{e}_i \in \mathrm{TM}_{\mathbf{X}}$ and $\mathbf{w} = w^\alpha \mathbf{f}_\alpha \in \mathrm{TN}_{\mathbf{x}}$, with $w^\alpha = v^i F^\alpha_i$.

The map $f_*^{\mathrm{T}} : \mathrm{TN}_{\mathbf{x}} \longrightarrow \mathrm{TM}_{\mathbf{X}}$ is induced from $f_* : \mathrm{TM}_{\mathbf{X}} \longrightarrow \mathrm{TN}_{\mathbf{x}}$ via transposition:

$$(f_*(\mathbf{v})|\mathbf{w})_{\mathrm{TN}_{\mathbf{x}}} = \left(\mathbf{v}|f_*^{\mathrm{T}}(\mathbf{w})\right)_{\mathrm{TM}_{\mathbf{X}}}\,,$$

in which $(\,\cdot\,|\,\cdot\,)$ denotes the inner product on the tangent space indicated by the attached subscript.

The Euler-Lagrange strain tensor E is defined as the mapping

$$E = \frac{1}{2}\left(f_*^{\mathrm{T}} \circ f_* - \mathrm{id}_{\mathrm{TM}_{\mathbf{X}}}\right) : \mathrm{TM}_{\mathbf{X}} \longrightarrow \mathrm{TM}_{\mathbf{X}} : \mathbf{v} \mapsto E(\mathbf{v}) = v^i E^j_i \mathbf{e}_j\,.$$

(10) **a.** Show that the components of E relative to $\{\mathbf{e}_j\}$ are given by $E^i_j = \dfrac{1}{2}\left(g^{i\ell} h_{\alpha\beta} F^\alpha_\ell F^\beta_j - \delta^i_j\right)$.

[1] We adhere to common malpractice by using symbols $\mathbf{x}(\mathbf{X}, t_0) = \mathbf{X}$ and $\mathbf{x}(\mathbf{X}, t) = \mathbf{x}$ to denote either mappings or independent variables, depending on context. No confusion is likely to arise.

Set $\mathbf{v} = v^i \mathbf{e}_i \in \mathrm{TM}_{\mathbf{X}}$, then $f_*(\mathbf{v}) = v^i F^\alpha_i \mathbf{f}_\alpha \in \mathrm{TN}_{\mathbf{x}}$ by definition. If $\mathbf{w} = w^\alpha \mathbf{f}_\alpha \in \mathrm{TN}_{\mathbf{x}}$, then $f_*^{\mathrm{T}}(\mathbf{w}) = g^{ij} F^\alpha_j w^\beta h_{\alpha\beta} \mathbf{e}_i \in \mathrm{TM}_{\mathbf{X}}$. To see this, insert the decompositions $f_*(\mathbf{v}) = v^i F^\alpha_i \mathbf{f}_\alpha$ and $\mathbf{w} = w^\alpha \mathbf{f}_\alpha$ into the defining equation, $(f_*(\mathbf{v})|\mathbf{w})_{\mathrm{TN}_{\mathbf{x}}} = \left(\mathbf{v}|f_*^{\mathrm{T}}(\mathbf{w})\right)_{\mathrm{TM}_{\mathbf{X}}}$, and solve for the components u^i of $f_*^{\mathrm{T}}(\mathbf{w}) = u^i \mathbf{e}_i$. Taking into account that this must hold for all $\mathbf{v} \in \mathrm{TM}_{\mathbf{X}}$ this yields $u^i = g^{ij} F^\alpha_j w^\beta h_{\alpha\beta}$. Substituting $\mathbf{w}$ by $f_*(\mathbf{v})$ then finally yields $E(\mathbf{v}) = v^j E^i_j \mathbf{e}_j = \frac{1}{2}\left(u^i - v^i\right)\mathbf{e}_i = \frac{1}{2} v^j \left(g^{i\ell} h_{\alpha\beta} F^\alpha_\ell F^\beta_j - \delta^i_j\right)\mathbf{e}_i$.

An isometric deformation is a rigid body motion, which by definition preserves inner products, i.e. $(f_*(\mathbf{u})|f_*(\mathbf{v}))_{\mathrm{TN}_{\mathbf{x}}} = (\mathbf{u}|\mathbf{v})_{\mathrm{TM}_{\mathbf{X}}}$.

(10) **b.** Show that E vanishes identically under isometric deformations. Thus unlike the deformation tensor, the strain tensor captures genuinely nonrigid deformations.

An isometric deformation by definition preserves inner products, i.e. $f_*^{\mathrm{T}} \circ f_* = \mathrm{id}_{\mathrm{TM}_{\mathbf{X}}}$, whence $E = 0$ identically.

(10) **c.** If the bases $\{\mathbf{e}_i\}$ is a *coordinate basis*, i.e. $\mathbf{e}_i = \partial/\partial X^i$, and if $\{\mathbf{f}_\alpha\}$ is the corresponding coordinate basis *induced by the deformation map*, i.e. $\mathbf{f}_\alpha = \partial/\partial x^\alpha$ with $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, show that $g_{ij} = h_{\alpha\beta} F^\alpha_i F^\beta_j$, and hence that $E^i_j = 0$ *regardless of the deformation*. Can you explain this result in words?

By the chain rule we have $h_{\alpha\beta} \hat{\mathbf{f}}^\alpha \otimes \hat{\mathbf{f}}^\beta = h_{\alpha\beta} dx^\alpha \otimes dx^\beta = h_{\alpha\beta} \frac{\partial x^\alpha}{\partial X^i} \frac{\partial x^\beta}{\partial X^j} dX^i \otimes dX^j = h_{\alpha\beta} F^\alpha_i F^\beta_j dX^i \otimes dX^j = g_{ij} \hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}^j$ in corresponding points. If your yardsticks employed for measuring lengths and angles deform along with your deformable object, then no strains—material length changes—can be observed.

(10) **d.** *Vox populi* in tensor calculus books stipulates that if the holor of a tensor relative to any given basis vanishes, then it vanishes relative to any other basis, i.e. the tensor is trivial. However, in problem c we have $E^i_j = 0$, whereas in problem a we have $E^i_j \neq 0$ in general. Explain this paradox.

The conjecture that a tensor vanishes if its holor vanishes pertains to a *single* basis $\{\mathbf{e}_i\}$ of $\mathrm{TM}_{\mathbf{X}}$, respectively to a composite basis of outer products of $\{\mathbf{e}_i\}$ and its dual basis $\{\hat{\mathbf{e}}^i\}$.

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(**30**) **3.** IMAGE INTRINSIC BASIS.

In this problem we model a scalar image $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (x, y) \mapsto f(x, y)$ as a $C^1(\mathbb{R}^2)$-function on the Euclidean plane $\mathbb{R}^2$, interpreted as a vector space with standard inner product relative to global Cartesian coordinates $(x, y) \in \mathbb{R}^2$, which induce a global Cartesian basis $\{\partial_x, \partial_y\}$ of $\mathbb{R}^2$. The tangent space $\mathrm{T}\mathbb{R}^2{}_{(x,y)}$ at any given point $(x, y) \in \mathbb{R}^2$ is spanned by a "local copy" $\{\partial_x, \partial_y\}$ of this global basis.

We henceforth consider only *regular points* in the image, i.e. point at which $\nabla f \neq 0$. At a regular point we define a new covector basis intrinsically coupled to the local image gradient, viz.

$$\hat{\mathbf{e}}^1 = *df = \boldsymbol{\epsilon} \lrcorner \flat df \quad \text{and} \quad \hat{\mathbf{e}}^2 = df \,.$$

Here, $* : \bigwedge_1(\mathrm{T}\mathbb{R}^2{}_{(x,y)}) \to \bigwedge_1(\mathrm{T}\mathbb{R}^2{}_{(x,y)})$ is the Hodge star operator.

(10) **a.** Give the components of $\hat{\mathbf{e}}^1$ and $\hat{\mathbf{e}}^2$ relative to the local Cartesian cotangent basis $\{dx, dy\}$.

We have, respectively, $\hat{\mathbf{e}}^1 = *df = \boldsymbol{\epsilon} \lrcorner \flat df = -f_y dx + f_x dy$ and $\hat{\mathbf{e}}^2 = f_x dx + f_y dy$. Here, f_x and f_y are shorthands for the partial derivatives of f w.r.t. x and y.

(10) **b.** Give the components of the corresponding basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ dual to $\hat{\mathbf{e}}^1$ and $\hat{\mathbf{e}}^2$ relative to the local Cartesian tangent basis $\{\partial_x, \partial_y\}$.

We have $\mathbf{e}_1 = -\dfrac{f_y}{f_x^2+f_y^2}\partial_x + \dfrac{f_x}{f_x^2+f_y^2}\partial_y$, respectively $\mathbf{e}_2 = \dfrac{f_x}{f_x^2+f_y^2}\partial_x + \dfrac{f_y}{f_x^2+f_y^2}\partial_y$.

(10) **c.** Indicate explicitly how to compute the Christoffel symbols $\overline{\Gamma}^k_{ij}$ relative to the image intrinsic basis, assuming that the Christoffel symbols $\Gamma^k_{ij} = 0$ for the Cartesian basis. Do *not* compute these symbols.

We may use the defining formula for the Christoffel symbols in the image intrinsic basis, $\overline{\Gamma}^k_{ij} = \left\langle \hat{\mathbf{e}}^k, \nabla_{\mathbf{e}_j}\mathbf{e}_i \right\rangle$, together with the defining properties of the affine connection, which allow us to rewrite these into linear combinations of $\langle \hat{\mathbf{e}}^i, \mathbf{e}_i \rangle$. Explicit expressions can be obtained using the expressions derived under a and b. In the case at hand we may write

$$\mathbf{e}_i = a_i^j \partial_j \quad \text{and} \quad \hat{\mathbf{e}}^i = b_j^i dx^j \,,$$

with matrices $A = \{a_i^j\}_{i,j=1,2}$, $B = \{b_i^j\}_{i,j=1,2}$,

$$A = \begin{pmatrix} -\dfrac{f_y}{f_x^2+f_y^2} & \dfrac{f_x}{f_x^2+f_y^2} \\ \dfrac{f_x}{f_x^2+f_y^2} & \dfrac{f_y}{f_x^2+f_y^2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -f_y & f_x \\ f_x & f_y \end{pmatrix} .$$

The subscript is the row index. In this notation we have $\langle \hat{\mathbf{e}}^i, \mathbf{e}_j \rangle = a_j^\ell b_k^i \left\langle dx^k, \partial_\ell \right\rangle = a_j^\ell b_\ell^i = \delta_j^i$.

THE END