EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2F800. Date: Monday April 11, 2011. Time: 14h00–17h00. Place: HG 6.96

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- You may consult the course notes "Tensorrekening en Differentiaalmeetkunde (2F800)" by Jan de Graaf, and the draft notes "Tensor Calculus and Differential Geometry (2F800)" by Luc Florack without warranty.

(**20**) **1.** COFACTOR MATRIX.

The cofactor matrix of a square matrix A with entries $A_{ij} \in \mathbb{R}$, $(i, j = 1, \ldots, n)$, is the matrix $\tilde{A}$ defined in terms of its entries

$$\tilde{A}^{ij} = \frac{\partial \det A}{\partial A_{ij}} .$$

(5) **a.** Show that if A is symmetric, i.e. $A_{ij} = A_{ji}$, then also $\tilde{A}$ is symmetric.

We take A to be a symmetric 3×3 matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} .$$

(5) **b.** Give the explicit matrix form of $\tilde{A}$.

Let $G(x)$ be the Gram matrix corresponding to a Riemannian metric field at $x \in \mathbb{R}^n$. Its covariant components $g_{ij}(x)$ vary smoothly with x. The inverse $G^{-1}(x)$ has contravariant components $g^{ij}(x)$. Its determinant is indicated by $g(x) = \det G(x)$. By ∂_k we mean $\partial/\partial x^k$.

(5) **c.** Show that $\nabla(\ln \det G) = \text{tr}\,(G^{-1}\nabla G)$ by showing that $\partial_k \ln g = g^{ij}\partial_k g_{ij}$.

(5) **d.** Show that $\partial_k \ln g = 2\Gamma^i_{ik}$, in which Γ^i_{jk} are the Christoffel symbols corresponding to the Lévi-Civita connection.
(*Hint:* Metric coefficients are "covariantly constant".)

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(**20**) **2.** CANONICAL MOMENTUM.

Let $x^i(t) \in \mathbb{R}$ denote the i-th coordinate of a material point particle in n-dimensional Euclidean space, with $i = 1, \ldots, n$, at time $t \in \mathbb{R}$, and $\dot{x}^i(t) = dx^i(t)/dt$. We introduce the so-called Lagrangian

$$L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (x, \dot{x}) \mapsto L(x, \dot{x}) \,,$$

i.e. a real-valued function of the particle's position and velocity. We consider a spatial reparametrization of the form[1] $x^i : \mathbb{R}^n \to \mathbb{R} : \overline{x} \mapsto x^i(\overline{x})$, in terms of new independent variables $\overline{x} \in \mathbb{R}^n$. Via substitution, and using the chain rule, we then define the reparametrized Lagrangian $\overline{L}(\overline{x}, \dot{\overline{x}}) = L(x, \dot{x})$.

(5) **a.** Show that the components of the particle's velocity "transforms as a vector" by showing that

$$\dot{x}^i = \frac{\partial x^i}{\partial \overline{x}^j} \, \dot{\overline{x}}^j \,.$$

(Thus we may regard x^i as a function of $\overline{x}^j$, and $\dot{x}^i$ as a function of $\overline{x}^j$ and $\dot{\overline{x}}^k$, $i, j, k = 1, \ldots, n$.)

The components of the so-called canonical momentum relative to a coordinate basis are defined as

$$p_i = \frac{\partial L(x, \dot{x})}{\partial \dot{x}^i} \,.$$

Likewise we require that

$$\overline{p}_i = \frac{\partial \overline{L}(\overline{x}, \dot{\overline{x}})}{\partial \dot{\overline{x}}^i} \,.$$

(10) **b.** Show that the components of the canonical momentum "transforms as a covector" by showing that

$$\overline{p}_i = \frac{\partial x^j}{\partial \overline{x}^i} \, p_j \,.$$

(5) **c.** Show that $p_i \dot{x}^i = \overline{p}_i \dot{\overline{x}}^i$.

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[1] We adhere to common malpractice by using the same symbol x to denote both an independent variable as well as a mapping. It should be clear from the context which of these two interpretations is applicable.

(**20**) **3.** EXTERIOR DERIVATIVE.

Let $\omega^k(x) = \omega_{i_1 \ldots i_k}(x)\, dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \bigwedge_k(\mathbb{R}^n)$ be an antisymmetric covariant tensor field of rank k (k-form for brevity), with $x \in \mathbb{R}^n$. The tangent space $\mathrm{T}\mathbb{R}^n{}_y$ at any point $y \in \mathbb{R}^n$ may be identified with $\mathbb{R}^n$. The *exterior derivative* of ω is the $(k+1)$-form given by

$$d\omega^k(x) = d\omega_{i_1 \ldots i_k}(x) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \bigwedge\nolimits_{k+1}(\mathbb{R}^n)\,,$$

with $df(x) \stackrel{\text{def}}{=} \partial_i f(x)\, dx^i$ for a sufficiently smooth scalar field (or 0-form) $f : \mathbb{R}^n \to \mathbb{R}$.

(5) **a.** Show that $d\omega^n(x) = 0$ regardless the choice of $\omega^n(x) \in \bigwedge_n(\mathbb{R}^n)$.

(5) **b.** Show that $d^2\omega^k(x) \stackrel{\text{def}}{=} dd\omega^k(x) = 0$ regardless the choice of $\omega^k(x) \in \bigwedge_k(\mathbb{R}^n)$ and $k = 0, \ldots, n$.

We now consider $n = 3$, and abbreviate $x^1 = x$, $x^2 = y$, $x^3 = z$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$, etc.

(5) **c.** Give the general form of $\omega^1(x, y, z) \in \bigwedge_1(\mathbb{R}^3)$ and compute the explicit form of $d\omega^1(x, y, z)$.

(5) **d.** Give the general form of $\omega^2(x, y, z) \in \bigwedge_2(\mathbb{R}^3)$ and compute the explicit form of $d\omega^2(x, y, z)$.

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(**40**) **4.** CURVATURE OPERATOR.

In this problem M is an n-dimensional smooth manifold. For the sake of simplicity we identify it with $\mathbb{R}^n$, i.e. we de-emphasize the role of coordinate charts. Linear superposition acts pointwise, i.e. vector arguments and scalar multipliers are to be regarded as (local samples of) smooth functions of $x \in \mathrm{M}$.

For $\mathbf{v}, \mathbf{w} \in \mathrm{TM}$ the (antisymmetric) curvature operator is defined as $\mathscr{R}(\mathbf{v}, \mathbf{w}) = [\nabla_{\mathbf{v}}, \nabla_{\mathbf{w}}] - \nabla_{[\mathbf{v}, \mathbf{w}]}$.

For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathrm{TM}$, $\widetilde{\mathbf{z}} \in \mathrm{T}^*\mathrm{M}$, the Riemann curvature tensor is defined as $\mathbf{R}(\widetilde{\mathbf{z}}, \mathbf{u}, \mathbf{v}, \mathbf{w}) = \langle \widetilde{\mathbf{z}}\,, \mathscr{R}(\mathbf{v}, \mathbf{w})\mathbf{u} \rangle$.

(10) **a.** Let $\lambda : \mathrm{M} \to \mathbb{R}$ be a smooth scalar field. Show that $\mathscr{R}(\lambda\, \mathbf{v}, \mathbf{w}) = \lambda\, \mathscr{R}(\mathbf{v}, \mathbf{w})$.

(10) **b.** Given the result of the previous problem, show that $\mathscr{R}(\mathbf{v}, \lambda\, \mathbf{w}) = \lambda\, \mathscr{R}(\mathbf{v}, \mathbf{w})$.

(10) **c.** Show that $\mathscr{R}(\mathbf{v}, \mathbf{w})(\lambda\, \mathbf{u}) = \lambda\, \mathscr{R}(\mathbf{v}, \mathbf{w})\mathbf{u}$.
(*Hint:* Argue why it suffices to consider $\mathbf{v} = \partial_i$, $\mathbf{w} = \partial_j$.)

(10) **d.** Show that $\mathbf{R}(\widetilde{\mathbf{z}}, \mathbf{u}, \mathbf{v}, \mathbf{w})$ is quadrilinear.

THE END