# RE-EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2F800. Date: June 15, 2009. Time: 14h00–17h00. Place: HG 10.30 E.

- Write your name and student identification number on each paper.
- The exam consists of 5 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- You may consult the course notes "Tensorrekening en Differentiaalmeetkunde (2F800)" by Jan de Graaf, and the draft notes "Tensor Calculus and Differential Geometry (2F800)" by Luc Florack without warranty.

(**25**) **1.** ANTISYMMETRIC COTENSORS IN $\mathbb{R}^3$

Let $\mathscr{B}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of Euclidean 3-space $V = \mathbb{R}^3$. Let $\mathscr{B}(\bigwedge^p(V))$ denote the induced basis for $\bigwedge^p(V)$, i.e. the linear space of antisymmetric contravariant tensors of rank $p$ on $V$, and $\mathscr{B}(\bigvee^p(V))$ the induced basis for $\bigvee^p(V)$, i.e. the linear space of symmetric contravariant tensors of rank $p$ on $V$.

($12\frac{1}{2}$) **a.** Complete the table for $\mathscr{B}(\bigwedge^p(V))$ in the appendix by providing the induced bases and their dimensions.

Cf. appendix, first table.

($12\frac{1}{2}$) **b.** Complete the table for $\mathscr{B}(\bigvee^p(V))$ in the appendix by providing the induced bases and their dimensions.

Cf. appendix, second table.

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(**20**) **2.** THE "INFINITESIMAL VOLUME ELEMENT".

We consider an open subset $\Omega \subset \mathrm{M}$ of a Riemannian manifold M with a single smooth coordinate chart

$$\xi : \Omega \to \mathbb{R}^k : p \mapsto \xi(p) = (\xi^1(p), \ldots, \xi^k(p)) \in \mathbb{R}^k .$$

Furthermore we take $\Omega$ to be correspond to the open box $\xi(\Omega) = (a^1, b^1) \times \ldots \times (a^k, b^k) \subset \mathbb{R}^k$.

Recall the classical change of variables theorem from integral calculus.

**Theorem.** If $\xi = \xi(\overline{\xi})$ represents a coordinate transformation, then, using self-explanatory notation for the boundary values,

$$\int_{a^1}^{b^1} \cdots \int_{a^k}^{b^k} f(\xi^1, \ldots, \xi^k)\, d\xi^1 \ldots d\xi^k = \int_{\overline{a}^1}^{\overline{b}^1} \cdots \int_{\overline{a}^k}^{\overline{b}^k} \overline{f}(\overline{\xi}^1, \ldots, \overline{\xi}^k) \det \frac{\partial \xi}{\partial \overline{\xi}}\, d\overline{\xi}^1 \ldots d\overline{\xi}^k ,$$

in which

$$\overline{f}(\overline{\xi}^1, \ldots, \overline{\xi}^k) = f(\xi^1(\overline{\xi}^1, \ldots, \overline{\xi}^k), \ldots, \xi^k(\overline{\xi}^1, \ldots, \overline{\xi}^k)) ,$$

and

$$\det \frac{\partial \xi}{\partial \overline{\xi}} \stackrel{\text{def}}{=} \det \begin{pmatrix} \frac{\partial \xi^1}{\partial \overline{\xi}^1} & \dots & \frac{\partial \xi^1}{\partial \overline{\xi}^n} \\ \dots & & \dots \\ \frac{\partial \xi^n}{\partial \overline{\xi}^1} & \dots & \frac{\partial \xi^n}{\partial \overline{\xi}^n} \end{pmatrix} .$$

Also recall the following "tensor transformation law".

**Lemma.** Let $T \in \mathbf{T}^p_q(V)$ be a mixed tensor, $\{\mathbf{e}_i\}$ a basis of the $n$-dimensional real linear space $V$, $\mathbf{f}_j = A^i_j \mathbf{e}_i$ a change of basis, with transformation matrix $A$, and $B = A^{-1}$, i.e. $A^i_k B^k_j = \delta^i_j$. Then

$$T = T^{i_1 \dots i_p}_{j_1 \dots j_q} \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_p} \otimes \hat{\mathbf{e}}^{j_1} \otimes \dots \otimes \hat{\mathbf{e}}^{j_q} \stackrel{\text{def}}{=} \overline{T}^{i_1 \dots i_p}_{j_1 \dots j_q} \mathbf{f}_{i_1} \otimes \dots \otimes \mathbf{f}_{i_p} \otimes \hat{\mathbf{f}}^{j_1} \otimes \dots \otimes \hat{\mathbf{f}}^{j_q} ,$$

if and only if the holor adheres to the tensor transformation law,

$$\overline{T}^{i_1 \dots i_p}_{j_1 \dots j_q} = A^{\ell_1}_{j_1} \dots A^{\ell_q}_{j_q} B^{i_1}_{k_1} \dots B^{i_p}_{k_p} T^{k_1 \dots k_p}_{\ell_1 \dots \ell_q} .$$

A scalar is defined as a tensor of type $(p, q) = (0, 0)$.

(5) **a.** Argue that the "infinitesimal volume element" $d\xi^1 \dots d\xi^k$ does not formally transform as a scalar, and derive the correct "relative scalar transformation law" to which it adheres.

From the theorem it is apparent that

$$d\overline{\xi}^1 \dots d\overline{\xi}^k = \det{}^{-1} \frac{\partial \xi}{\partial \overline{\xi}} \, d\xi^1 \dots d\xi^k .$$

The appearance of the Jacobian determinant violates the formal scalar transformation rule.

If $\mathbf{v}(p) = v^i(\xi(p)) \partial_i|_p \in \mathrm{TM}_p$, $\mathbf{w}(p) = w^i(\xi(p)) \partial_i|_p \in \mathrm{TM}_p$, in which $\mathrm{TM}_p$ is the tangent space to M at $p \in \Omega$, then we indicate the inner product on $\mathrm{TM}_p$ by $G(p)(\xi, \eta) = (\xi|\eta)_p = g_{ij}(\xi(p)) v^i(\xi(p)) w^j(\xi(p))$, in which $g_{ij}(\xi(p))$ is the (holor of the) metric tensor at $p \in \Omega$.

For notational simplicity we suppress the explicit base point label $p$ and its reference coordinates $\xi(p)$ in the above notation, and simply write $\mathbf{v} = v^i \partial_i$, $\mathbf{w} = w^i \partial_i$, $G(\xi, \eta) = (\xi|\eta) = g_{ij} v^i w^j$, et cetera. The determinant of the matrix $g_{ij}$ is abbreviated as $g$.

(5) **b.** Show that the "invariant infinitesimal volume element", $dV \stackrel{\text{def}}{=} \sqrt{g}\, d\xi^1 \dots d\xi^k$, transforms as a "pseudo scalar", i.e. in a way identical to a scalar up to a possible minus sign: When does the minus sign arise?

Using

$$\overline{g} = \left( \det \frac{\partial \xi}{\partial \overline{\xi}} \right)^2 g \quad \text{whence} \quad \sqrt{\overline{g}} = \left| \det \frac{\partial \xi}{\partial \overline{\xi}} \right| \sqrt{g} ,$$

and the change of variables theorem, we observe that

$$d\overline{V} \stackrel{\text{def}}{=} \sqrt{\overline{g}}\, d\overline{\xi}^1 \dots d\overline{\xi}^k = |\det \frac{\partial \xi}{\partial \overline{\xi}}| \sqrt{g} \det \frac{\partial \overline{\xi}}{\partial \xi} \, d\xi^1 \dots d\xi^k = \operatorname{sgn} \left( \det \frac{\partial \xi}{\partial \overline{\xi}} \right) \sqrt{g}\, d\xi^1 \dots d\xi^k = \operatorname{sgn} \left( \det \frac{\partial \xi}{\partial \overline{\xi}} \right) dV .$$

A minus sign shows up if the orientation of the coordinate system changes.

(5) **c.** Show that the Levi-Civita tensor, $\boldsymbol{\epsilon} = \sqrt{g}\, d\xi^1 \wedge \dots \wedge d\xi^k = \epsilon_{|\mu_1 \dots \mu_k|} d\xi^{\mu_1} \wedge \dots \wedge d\xi^{\mu_k} \in \bigwedge_k(\mathrm{TM}_p)$,

with $\epsilon_{1\ldots k} = \sqrt{g}$, formally transforms in the same way as $dV$.

We have

$$\begin{aligned}
\overline{\boldsymbol{\epsilon}} &= \sqrt{\overline{g}}\, d\overline{\xi}^1 \wedge \ldots \wedge d\overline{\xi}^k \stackrel{\mathrm{b}}{=} \left|\det \frac{\partial \xi}{\partial \overline{\xi}}\right| \sqrt{g} \left(\frac{\partial \overline{\xi}^1}{\partial \xi^{i_1}} d\xi^{i_1}\right) \wedge \ldots \wedge \left(\frac{\partial \overline{\xi}^k}{\partial \xi^{i_k}} d\xi^{i_k}\right) = \left|\det \frac{\partial \xi}{\partial \overline{\xi}}\right| \sqrt{g}\, \frac{\partial \overline{\xi}^1}{\partial \xi^{i_1}} \ldots \frac{\partial \overline{\xi}^k}{\partial \xi^{i_k}}\, d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_k} \\
&= \left|\det \frac{\partial \xi}{\partial \overline{\xi}}\right| \sqrt{g}\, \frac{\partial \overline{\xi}^1}{\partial \xi^{i_1}} \ldots \frac{\partial \overline{\xi}^k}{\partial \xi^{i_k}}\, [i_1 \ldots i_k]\, d\xi^1 \wedge \ldots \wedge d\xi^k \stackrel{*}{=} \operatorname{sgn} \det \frac{\partial \xi}{\partial \overline{\xi}} \sqrt{g}\, d\xi^1 \wedge \ldots \wedge d\xi^k = \operatorname{sgn} \det \frac{\partial \xi}{\partial \overline{\xi}}\, \boldsymbol{\epsilon}\,.
\end{aligned}$$

In $*$ we have made use of $\dfrac{\partial \overline{\xi}^1}{\partial \xi^{i_1}} \ldots \dfrac{\partial \overline{\xi}^k}{\partial \xi^{i_k}}\, [i_1 \ldots i_k] = \det \dfrac{\partial \overline{\xi}}{\partial \xi}$ and of $\det \dfrac{\partial \overline{\xi}}{\partial \xi} = \det^{-1} \dfrac{\partial \xi}{\partial \overline{\xi}}$.

The Levi-Civita (or metric) connection provides a covariant derivative operator $\nabla$, with formal components $D_j$ relative to a coordinate basis $\{dx^j\}$. Recall that for the contravariant components of a vector field we have $D_j v^k = \partial_j v^k + \Gamma^k_{ij} v^i$, with

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\partial_i g_{\ell j} + \partial_j g_{\ell i} - \partial_\ell g_{ij}\right)\,.$$

(5) **d.** Show that if $\hat{\mathbf{v}} = v_i dx^i \in \mathrm{T}^*\mathrm{M}$ and $\mathbf{w} = w^i \partial_i \in \mathrm{TM}$ are smooth covector, respectively vector fields "of rapid decay", i.e. such that all derivatives of the component functions $v_i : \mathbb{R}^k \to \mathbb{R} : q \mapsto v_i(q)$ and $w^i : \mathbb{R}^k \to \mathbb{R} : q \mapsto w^i(q)$ vanish in the limit $\|q\| \to \infty$, then

$$\int_{\mathbb{R}^k} v_i(\xi) \left(D_j w^i(\xi)\right) \sqrt{g}(\xi)\, d\xi^1 \ldots d\xi^k = -\int_{\mathbb{R}^k} \left(D_j v_i(\xi)\right) w^i(\xi) \sqrt{g}(\xi)\, d\xi^1 \ldots d\xi^k\,.$$

By construction of the Levi-Civita connection the metric is "covariantly constant", i.e. $D_k g_{ij} = 0$, whence also $D_k \sqrt{g} = 0$. Therefore

$$D_j \left(v_i w^i \sqrt{g}\right) = \left(D_j v_i w^i + v_i D_j w^i\right) \sqrt{g}\,,$$

from which the result follows by integration, taking into account that integration of the left hand side yields zero by virtue of rapid decay.

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(**25**) **3.** CARTESIAN TENSORS

Let M be a smooth manifold furnished with an atlas of coordinate charts, and consider a coordinate transformation $y^i = y^i(x^1, \ldots, x^n)$, $i = 1, \ldots, n$. (For simplicity we write $y^i = y^i(x^j)$ henceforth.) The Jacobian matrix is indicated by $\mathbf{A}$, with components $A^i_j = \partial y^i / \partial x^j$, and its inverse by $\mathbf{B}$, with $B^i_j = \partial x^i / \partial y^j$. (The upper index acts as the row index.)

For $\mathbf{p} \in$ M, $\{\partial_{x^i}\}$ denotes the coordinate basis of the local tangent space $\mathrm{TM}_\mathbf{p}$ in $x$-coordinatization. An arbitrary vector can be decomposed as $\mathbf{v} = v^i \partial_{x^i}$ relative to $\{\partial_{x^i}\}$. The cotangent space $\mathrm{T^*M}_\mathbf{p}$ is spanned by $\{dx^i\}$, and an arbitrary covector can be written as $\hat{\mathbf{v}} = v_i dx^i$ relative to $\{dx^i\}$.

(5) **a.** Show that if $\mathbf{v} = w^i \partial_{y^i}$ and $\hat{\mathbf{v}} = w_i dy^i$, then $w^i = A^i_j v^j$, respectively $w_i = B^j_i v_j$.

We have $\mathbf{v} = v^j \partial_{x^j} = v^j A^i_j \partial_{y^i} = w^i \partial_{y^i}$, from which we infer that $w^i = A^i_j v^j$. Likewise, $\hat{\mathbf{v}} = v_j dx^j = v_j B^j_i dy^i = w_i dy^i$, whence $w_i = B^j_i v_j$.

We impose a Riemannian structure on M by stipulating an inner product (on each tangent space). If $\mathbf{v}, \mathbf{w} \in \mathrm{TM}_\mathbf{p}$, then $(\mathbf{v}|\mathbf{w}) = G(\mathbf{v}, \mathbf{w}) = g_{ij} v^i w^j$, in which $g_{ij} = (\partial_{x^i} | \partial_{x^j})$ are the components of the corresponding Gram matrix $\mathbf{G}$ relative to $\{\partial_{x^i}\}$. (Notice the distinction between the coordinate independent rank-2 cotensor $G$ and its coordinate dependent matrix representation $\mathbf{G}$.)

(5) **b.** Show that if $G = g_{ij} dx^i \otimes dx^j = h_{ij} dy^i \otimes dy^j$, in which $h_{ij}$ are the components of the Gram matrix $\mathbf{H}$ in $y$-coordinatization, then $\mathbf{H} = \mathbf{B}^\mathrm{T} \mathbf{G} \mathbf{B}$.

We have $G = g_{ij} dx^i \otimes dx^j = g_{ij} B^i_k B^j_\ell dy^k \otimes dy^\ell$, whence it follows that $h_{k\ell} = B^i_k B^j_\ell g_{ij}$. In matrix notation, identifying upper (lower) or leftmost (rightmost) indices as row (column) indices, this can be expressed as $\mathbf{H} = \mathbf{B}^\mathrm{T} \mathbf{G} \mathbf{B}$.

**Definition.** The converters $\sharp : \mathrm{TM}_\mathbf{p} \to \mathrm{T^*M}_\mathbf{p}$ and $\flat : \mathrm{T^*M}_\mathbf{p} \to \mathrm{TM}_\mathbf{p}$ are defined as follows. If $\mathbf{v} \in \mathrm{TM}_\mathbf{p}$, $\hat{\mathbf{v}} \in \mathrm{T^*M}_\mathbf{p}$, then

$$\sharp \mathbf{v} \stackrel{\text{def}}{=} G(\mathbf{v}, \cdot\,) \in \mathrm{T^*M}_\mathbf{p} \quad \text{respectively} \quad \flat \hat{\mathbf{v}} \stackrel{\text{def}}{=} G^{-1}(\hat{\mathbf{v}}, \cdot\,) \in \mathrm{TM}_\mathbf{p}\,.$$

The dual metric $G^{-1}$ is the rank-2 contratensor uniquely defined in terms of $G$ by the requirement that the converters be eachother's inverse:

$$\sharp \circ \flat = \mathrm{id}_{\mathrm{T^*M}_\mathbf{p}} \quad \text{respectively} \quad \flat \circ \sharp = \mathrm{id}_{\mathrm{TM}_\mathbf{p}}\,.$$

(5) **c.** Show that $G^{-1} = g^{ij} \partial_{x^i} \otimes \partial_{x^j}$, in which $g^{ij}$ are the components of the inverse Gram matrix $\mathbf{G}^{-1}$. (*Hint:* Work out $(\flat \circ \sharp)\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathrm{TM}_\mathbf{p}$ in coordinate form.)

Suppose $\flat dx^j = \gamma^{jk} \partial_{x^k}$ for some $\gamma^{jk}$, which is possible by definition of the prototype of the operator $\flat$. We then have $\flat \circ \sharp \mathbf{v} = \flat G(\mathbf{v}, \cdot\,) = \flat g_{ij} v^i dx^j = g_{ij} v^i \flat dx^j = g_{ij} v^i \gamma^{jk} \partial_{x^k} \stackrel{\text{def}}{=} v^k \partial_{x^k} = \mathbf{v}$. Since this must hold for all $\mathbf{v} \in \mathrm{TM}_\mathbf{p}$ it follows that $g_{ij} \gamma^{jk} = \delta^k_i$, in other words, $\gamma^{ij} = g^{ij}$.

We now assume that M is a Euclidean space, and that, in $x$-coordinatization, $\mathbf{G} = \mathbf{I}$, the identity matrix (i.e. $x^i$ are Cartesian coordinates, and $\{\partial_{x^i}\}$ is an orthonormal basis of $\mathrm{TM}_\mathbf{p}$).

**Definition.** A coordinate transformation relating Cartesian coordinate systems is called a Cartesian coordinate transformation.

(5) **d.** Give the explicit form of an arbitrary Cartesian coordinate transformation $y^i = y^i(x^j)$.

The defining equation for Cartesian coordinate systems $x$ and $y$ is $\mathbf{G} = \mathbf{H} = \mathbf{I}$. According to b we thus have in $y$-coordinatization $\mathbf{I} = \mathbf{B}^{\mathrm{T}}\mathbf{B}$. Consequently $\mathbf{B}$, and hence also its inverse $\mathbf{A}$, must be an orthogonal matrix. This implies that $y^i = a^i + A^i_j x^j$ (with $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$).

(5) **e.** Argue why in this case no distinction needs to be made between a vector $\mathbf{v} \in \mathrm{TM}_{\mathbf{p}}$ and its dual $\hat{\mathbf{v}} = \sharp\mathbf{v} \in \mathrm{T}^*\mathrm{M}_{\mathbf{p}}$, or between upper and lower indices.

If we rewrite the transformation laws for vector and covector components, recall **a**, in matrix-vector form, we get

$$\begin{pmatrix} w^1 \\ \vdots \\ w^n \end{pmatrix} = A \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \quad \text{respectively} \quad \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = B^{\mathrm{T}} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = B^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} .$$

Thus the contravariant components $v^i$ of $\mathbf{v} \in \mathrm{TM}$ and the covariant components $v_i$ of $\hat{\mathbf{v}} \in \mathrm{T}^*\mathrm{M}$ transform in the same way. In addition, since $v^i = g^{ij}v_j$, contravariant and covariant components coincide numerically by virtue of the fact that $\mathbf{G} = \mathbf{I}$ in any Cartesian coordinate system. As a result, no confusion is likely to arise if we label all components by lower indices only.

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(**15**) **4.** GEODESIC COORDINATES

**Definition.** The Christoffel symbols $\Gamma^k_{ij}$ associated with the affine connection $\nabla : \mathrm{T}\mathbb{R}^n \times \mathrm{T}\mathbb{R}^n \to \mathrm{T}\mathbb{R}^n$ are defined by $\nabla_{\partial_j}\partial_i = \Gamma^k_{ij}\partial_k$ $(i, j, k = 1, \ldots, n)$.

Recall that the components of the Christoffel symbols in a coordinate system $\overline{\mathbf{x}}$, say $\overline{\Gamma}^k_{ij}$, are given relative to those in a coordinate system $\mathbf{x}$, i.e. $\Gamma^k_{ij}$, by

$$\overline{\Gamma}^k_{ij} = S^k_\ell \left( T^m_j T^n_i \Gamma^\ell_{nm} + \overline{\partial}_j T^\ell_i \right) ,$$

in which $\mathbf{T}$ and $\mathbf{S}$ are the Jacobian matrices with components

$$T^i_j = \frac{\partial x^i}{\partial \overline{x}^j} \quad \text{resp.} \quad S^i_j = \frac{\partial \overline{x}^i}{\partial x^j} .$$

• Let $\Gamma^k_{ij}(\mathscr{P})$ be given at a *fixed* point $\mathscr{P}$, with coordinates $x^i(\mathscr{P}) = a^i$, say. Show that there exists a coordinate system $\overline{\mathbf{x}}$ such that $\overline{x}^i(\mathscr{P}) = 0$ and $\overline{\Gamma}^k_{ij}(\mathscr{P}) = 0$.
(*Hint:* Postulate a transformation of the type $x^i = \alpha^i + \overline{x}^i + \frac{1}{2}\alpha^i_{jk}\overline{x}^j\overline{x}^k$.)

Obviously we must set $\alpha^i = a^i$. We have furthermore

$$T^i_j = \frac{\partial x^i}{\partial \overline{x}^j} = \delta^i_j + \alpha^i_{jk}\overline{x}^k \quad \text{and} \quad \overline{\partial}_j T^k_i = \alpha^k_{ij} ,$$

so in particular

$$T^i_j(\mathscr{P}) = \delta^i_j \quad \text{and} \quad \overline{\partial}_j T^k_i(\mathscr{P}) = \alpha^k_{ij} .$$

Substitution into the equation $\overline{\Gamma}^k_{ij}(\mathscr{P}) = 0$, i.e.

$$T^m_j(\mathscr{P}) T^n_i(\mathscr{P}) \Gamma^k_{nm}(\mathscr{P}) + \overline{\partial}_j T^k_i(\mathscr{P}) = 0 ,$$

yields $\Gamma^k_{ij}(\mathscr{P}) + \alpha^k_{ij} = 0$, whence $\alpha^k_{ij} = -\Gamma^k_{ij}(\mathscr{P})$.

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**(15)** **5.** CROSS PRODUCT IN $\mathbb{R}^3$

Given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $V = \mathbb{R}^3$ and a corresponding decomposition $\mathbf{v} = v^i \partial_i$, $\mathbf{w} = w^i \partial_i$, an inner product on $V$ is specified by $(\mathbf{v}|\mathbf{w}) = G(\mathbf{v}, \mathbf{w}) = g_{ij} v^i w^j$, in which $g_{ij} = (\mathbf{e}_i|\mathbf{e}_j)$ are the components of the corresponding Gram matrix $\mathbf{G}$. The dual basis of $V^*$ is indicated by $\{\hat{\mathbf{e}}^1, \hat{\mathbf{e}}^2, \hat{\mathbf{e}}^3\}$.

**Definition.** Let $\hat{\mathbf{a}}^1, \ldots, \hat{\mathbf{a}}^k \in V^*$. The Hodge star operator $* : \bigwedge_k(V) \to \bigwedge_{n-k}(V)$ is defined as follows:

$$\begin{cases} *1 & = \boldsymbol{\epsilon} & \text{for } k = 0, \\ * \left( \hat{\mathbf{a}}^1 \wedge \ldots \wedge \hat{\mathbf{a}}^k \right) & = \boldsymbol{\epsilon} \lrcorner \flat \hat{\mathbf{a}}^1 \lrcorner \ldots \lrcorner \flat \hat{\mathbf{a}}^k & \text{for } k = 1, \ldots, n. \end{cases}$$

In this definition $\boldsymbol{\epsilon} = \sqrt{g}\, \hat{\mathbf{e}}^1 \wedge \hat{\mathbf{e}}^2 \wedge \hat{\mathbf{e}}^3 = \epsilon_{|i_1 i_2 i_3|} \hat{\mathbf{e}}^{i_1} \wedge \hat{\mathbf{e}}^{i_2} \wedge \hat{\mathbf{e}}^{i_3} \in \bigwedge_3(V)$ is the Levi-Civita tensor, with $\epsilon_{123} = \sqrt{g} = \sqrt{\det \mathbf{G}}$.

**Definition.** Let $\mathbf{a}_1, \ldots, \mathbf{a}_k \in V$. The $(n-k)$-form $\boldsymbol{\epsilon} \lrcorner \mathbf{a}_1 \lrcorner \ldots \lrcorner \mathbf{a}_k \in \bigwedge_{n-k}(V)$ is defined as follows:

$$(\boldsymbol{\epsilon} \lrcorner \mathbf{a}_1 \lrcorner \ldots \lrcorner \mathbf{a}_k)(\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n) = \boldsymbol{\epsilon}(\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n) \quad \text{for all } \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n \in V.$$

**Definition** The cross product $\times : V \times V \to V : (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \times \mathbf{w}$ is defined in terms of the Levi-Civita tensor and Hodge star operator as follows:

$$\mathbf{v} \times \mathbf{w} = \flat \left( * \left( \sharp \mathbf{v} \wedge \sharp \mathbf{w} \right) \right) .$$

Recall problem 3 for the definition of $\sharp$ and $\flat$.

• Show that the components of $\mathbf{u} = \mathbf{v} \times \mathbf{w}$ are given by $u^\ell = \epsilon_{ijk} v^i w^j g^{k\ell}$. Give the explicit formula for $(u^1, u^2, u^3)$ in case of an orthonormal basis.

We have

$$\mathbf{v} \times \mathbf{w} = \flat \left( * \left( \sharp \mathbf{v} \wedge \sharp \mathbf{w} \right) \right) = \flat \left( \boldsymbol{\epsilon} \lrcorner (\flat \circ \sharp) \mathbf{v} \lrcorner (\flat \circ \sharp) \mathbf{w} \right) = \flat \left( \boldsymbol{\epsilon} \lrcorner \mathbf{v} \lrcorner \mathbf{w} \right) = \flat \left( \epsilon_{ijk} v^i w^j \hat{\mathbf{e}}^k \right) = \epsilon_{ijk} v^i w^j g^{k\ell} \mathbf{e}_\ell \,,$$

from which the result follows. If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is orthonormal, then $\mathbf{G} = \mathbf{I}$ and $\epsilon_{ijk} = [ijk]$, whence $\mathbf{u} = \mathbf{v} \times \mathbf{w} = [ijk] v^i w^j \mathbf{e}_k$, or

$$\begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \times \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix} = \begin{pmatrix} v^2 w^3 - v^3 w^2 \\ v^3 w^1 - v^1 w^3 \\ v^1 w^2 - v^2 w^1 \end{pmatrix} .$$

**THE END**

# APPENDIX

Name: Student number:

| $p$ | $\mathscr{B}(\bigwedge^p(V))$ | dim |
|---|---|---|
| 0 | $\{1\}$ | 1 |
| 1 | $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ | 3 |
| 2 | $\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3\}$ | 3 |
| 3 | $\{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\}$ | 1 |

| $p$ | $\mathscr{B}(\bigvee^p(V))$ | dim |
|---|---|---|
| 0 | $\{1\}$ | 1 |
| 1 | $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ | 3 |
| 2 | $\{\mathbf{e}_1 \vee \mathbf{e}_1, \mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{e}_1 \vee \mathbf{e}_3, \mathbf{e}_2 \vee \mathbf{e}_2, \mathbf{e}_2 \vee \mathbf{e}_3, \mathbf{e}_3 \vee \mathbf{e}_3\}$ | 6 |
| 3 | $\{\mathbf{e}_1 \vee \mathbf{e}_1 \vee \mathbf{e}_1, \mathbf{e}_1 \vee \mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{e}_1 \vee \mathbf{e}_1 \vee \mathbf{e}_3, \mathbf{e}_1 \vee \mathbf{e}_2 \vee \mathbf{e}_2, \mathbf{e}_1 \vee \mathbf{e}_2 \vee \mathbf{e}_3, \mathbf{e}_1 \vee \mathbf{e}_3 \vee \mathbf{e}_3, \mathbf{e}_2 \vee \mathbf{e}_2 \vee \mathbf{e}_2, \mathbf{e}_2 \vee \mathbf{e}_2 \vee \mathbf{e}_3, \mathbf{e}_2 \vee \mathbf{e}_3 \vee \mathbf{e}_3, \mathbf{e}_3 \vee \mathbf{e}_3 \vee \mathbf{e}_3\}$ | 10 |