EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2F800. Date: Wednesday April 17, 2013. Time: 14h00–17h00. Place: AUD 15

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- You may consult the course notes "Tensorrekening en Differentiaalmeetkunde (2F800)" by Jan de Graaf, and the draft notes "Tensor Calculus and Differential Geometry (2F800)" by Luc Florack without warranty.

(**30**) **1.** HELICOID.

We consider a 2-dimensional smooth surface S embedded in Euclidean 3-space $\mathbb{R}^3$, parametrized by

$$\Phi : \mathrm{S} \to \mathbb{R}^3 : (\xi, \eta) \mapsto (x(\xi, \eta), y(\xi, \eta), z(\xi, \eta)) \ .$$

The embedding 3-space is furnished with an inner product (or Riemannian metric)

$$G : \mathrm{T}\mathbb{R}^3 \times \mathrm{T}\mathbb{R}^3 \to \mathbb{R} : (v, w) \mapsto G(v, w) = (v|w) \ .$$

We define the *pull back* Φ^*G of G under Φ as the mapping

$$\Phi^*G : \mathrm{TS} \times \mathrm{TS} \to \mathbb{R} : (u, v) \mapsto \Phi^*G(u, v) \stackrel{\text{def}}{=} G(d\Phi(u), d\Phi(v)) = (d\Phi(u)|d\Phi(v)) \ ,$$

in which the differential of the parametrization map is defined as

$$d\Phi \stackrel{\text{def}}{=} \frac{\partial x^i}{\partial \xi^\alpha} d\xi^\alpha \otimes \partial_i \in \mathrm{T}^*\mathrm{S} \times \mathrm{T}\mathbb{R}^3 \quad \text{so that for } u \in \mathrm{TS} \quad d\Phi(u) = \frac{\partial x^i}{\partial \xi^\alpha} u^\alpha \partial_i \in \mathrm{T}\mathbb{R}^3 \, .$$

For notational simplicity we furthermore define the Jacobian matrix A, with components

$$A^i_\alpha = \frac{\partial x^i}{\partial \xi^\alpha} \quad (i \text{ and } \alpha \text{ are row and column index of A, respectively}) \, .$$

You may assume that A has everywhere matrix-rank 2. Notation: $x^1 = x, x^2 = y, x^3 = z$, $\xi^1 = \xi, \xi^2 = \eta$. Symbolic components relative to (co)tangent bases of $\mathbb{R}^3$ are denoted by Latin indices from the middle of the alphabet $(i, j, k, \ldots)$, while those pertaining to (co)tangent bases of S are indicated with Greek indices from the beginning of the alphabet $(\alpha, \beta, \gamma, \ldots)$.

($7\frac{1}{2}$) **a.** Show that Φ^*G defines an inner product on the embedded surface S.

Let $u, v, w \in \mathrm{TS}$ and $\lambda, \mu \in \mathbb{R}$.

- Symmetry: $\Phi^*G(u, v) = G(d\Phi(u), d\Phi(v)) \stackrel{*}{=} G(d\Phi(v), d\Phi(u)) = \Phi^*G(v, u)$.

- Linearity: $\Phi^*G(\lambda u+\mu v,w) = G(d\Phi(\lambda u+\mu v),d\Phi(w)) \stackrel{\circ}{=} G(\lambda d\Phi(u)+\mu d\Phi(v),d\Phi(w)) \stackrel{*}{=} \lambda G(d\Phi(u),d\Phi(w)) + \mu G(d\Phi(v),d\Phi(w)) = \lambda\Phi^*G(u,w)+\mu\Phi^*G(v,w)$.

- Positivity and nondegeneracy: $\Phi^*G(u,u) = G(d\Phi(u),d\Phi(u)) \stackrel{*}{\geq} 0$. Equality implies $d\Phi(u)=0$, or $Au=0$ in matrix notation. Since A has full rank (viz. 2) this implies $u=0$.

The (in)equalities marked by $*$ rely on the fact that G is an inner product. Those marked by $\circ$ exploit linearity of $d\Phi$.

Let g_{ij} be the covariant holor of the metric tensor G, and $\gamma_{\alpha\beta}$ that of the induced metric tensor Φ^*G.

$(7\frac{1}{2})$ **b.** Express $\gamma_{\alpha\beta}$ in terms of g_{ij}.

From $d\Phi(u^\alpha\partial_\alpha) = A^i_\alpha\, u^\alpha\partial_i$ it follows that $d\Phi(\partial_\alpha) = A^i_\alpha\partial_i$. Using this and $g_{ij} = G(\partial_i,\partial_j)$ we get $\gamma_{\alpha\beta} = \Phi^*G(\partial_\alpha,\partial_\beta) = G(d\Phi(\partial_\alpha),d\Phi(\partial_\beta)) = A^i_\alpha g_{ij} A^j_\beta$.

Henceforth we assume that G is the standard inner product, and the x^i are Cartesian coordinates, so that the holor of G is given by $g_{ij}=1$ if $i=j$ and $g_{ij}=0$ otherwise.

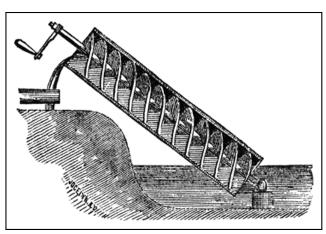
$(7\frac{1}{2})$ **c.** Show that $\gamma_{\alpha\beta} = \left(\mathrm{A}^\mathrm{T}\mathrm{A}\right)_{\alpha\beta}$, in which A^T is the transposed matrix of A.

The index expression in item b reads in matrix notation $\gamma_{\alpha\beta} = A^i_\alpha g_{ij} A^j_\beta = (\mathrm{A}^\mathrm{T} G\mathrm{A})_{\alpha\beta}$. Since G is the identity matrix the result follows.

We take S to be a helicoid ("Archimedean skrew"), with parameters $\xi=\xi^1=\rho\in\mathbb{R}$ and $\eta=\xi^2=\phi\in\mathbb{R}$:

$$\Phi : \mathrm{S}\to\mathbb{R}^3 : (\xi,\eta)\mapsto \left\{\begin{array}{lcl} x(\rho,\phi) & = & \rho\cos\phi \\ y(\rho,\phi) & = & \rho\sin\phi \\ z(\rho,\phi) & = & \phi \end{array}\right.$$

☞ A helicoid, discovered by Jean Baptiste Meusnier in 1776, is a so-called minimal surface bounded by a helix (in this problem we ignore the boundary). It has certain unique properties, e.g. it is the only ruled minimal surface other than the plane, meaning that it can be traced by a line, and the only non-rotary surface which can glide along itself. Helicoids are encountered in various engineering as well as natural devices, cf. the illustrations of a skrewpump and of the sophisticated myofibre architecture of the heart wall.

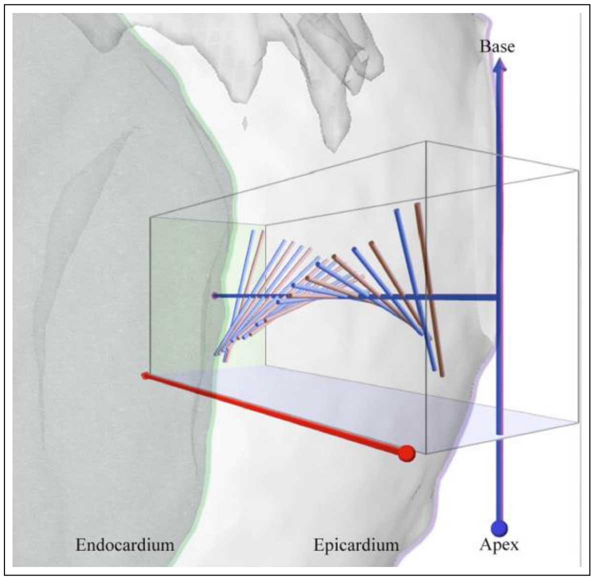


AN "ARCHIMEDEAN SKREW" TO DRAIN WATER FROM A LOW LYING AREA INTO AN IRRIGATION DITCH.

$(7\frac{1}{2})$ **d.** Compute the components $\gamma_{\alpha\beta}$ of the Gram matrix $\Gamma=\Phi^*G$.

Cf. item c. We find, expressed in (ρ,ϕ)-parameters,

$$\mathrm{A}^\mathrm{T}\mathrm{A} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 1 \end{pmatrix}\begin{pmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2+1 \end{pmatrix}$$ so $\gamma_{\rho\rho}=1$, $\gamma_{\rho\phi}=\gamma_{\phi\rho}=0$, and $\gamma_{\phi\phi}=\rho^2+1$.



P. Savadjiev, G. J. Strijkers, A. J. Bakermans, E. Piuze, S. W. Zucker, and K. Siddiqi.
"Heart wall myofibers are arranged in minimal surfaces to optimize organ function".
© Proc Natl Acad Sci USA, June 12, 2012, vol. 109, nr. 24, pp. 9248–9253.

(**30**) **2.** PSEUDO-RIEMANNIAN MANIFOLD.

Definition. A pseudo-Riemannian manifold is a manifold furnished with a pseudo-Riemannian metric $G : \mathrm{TM} \times \mathrm{TM} \to \mathbb{R}$, satisfying the following axioms (instead of $G((x, \mathbf{v}), (x, \mathbf{w}))$ we will write $G(x)(\mathbf{v}, \mathbf{w})$). For all points $x \in \mathrm{M}$, vectors $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathrm{TM}_x$, and scalar fields $\lambda, \mu \in C^\infty(\mathrm{M})$:

- $G(x)(\mathbf{v}, \mathbf{w}) = G(x)(\mathbf{w}, \mathbf{v})$,
- $G(x)(\lambda(x)\mathbf{v} + \mu(x)\mathbf{w}, \mathbf{z}) = \lambda(x)G(x)(\mathbf{v}, \mathbf{z}) + \mu(x)G(x)(\mathbf{w}, \mathbf{z})$,
- if $\mathbf{v} \neq \mathbf{0} \in \mathrm{TM}_x$ then there exists a $\mathbf{w} \in \mathrm{TM}_x$ such that $G(x)(\mathbf{v}, \mathbf{w}) \neq 0$ (“non-degeneracy”).

Given a basis $\{\partial_i\}$ of TM_x and a decomposition $\mathbf{v} = v^i(x)\partial_i$, $\mathbf{w} = w^i(x)\partial_i$, we may write

$$G(x)(\mathbf{v}, \mathbf{w}) = g_{ij}(x)v^i(x)w^j(x)\,,$$

with $g_{ij}(x) = G(x)(\partial_i, \partial_j)$, the holor of $G(x)$. The dual basis of $\mathrm{T}^*\mathrm{M}_x$ is indicated by $\{dx^i\}$.

(10) **a.** Show that $g(x) \stackrel{\text{def}}{=} \det g_{ij}(x)$ is not necessarily positive, but that it cannot be zero.
(*Hint:* Recall that a real symmetric matrix has real eigenvalues, and consider the non-degeneracy condition.)

Let $x \in \mathrm{M}$ be any fixed point. For all nonzero $\mathbf{v} = v^i(x)\partial_i \in \mathrm{TM}_x$ there exists a $\mathbf{w} = w^i(x)\partial_i \in \mathrm{TM}_x$ such that $g_{ij}(x)v^i(x)w^j(x) \neq 0$. This implies that $\hat{\mathbf{v}} = g_{ij}(x)v^j(x)dx^i \in \mathrm{T}^*\mathrm{M}_x$ is a nonzero covector. In turn this implies that the (symmetric) matrix with components $g_{ij}(x)$ does not have a null eigenvalue. However, it may have positive as well as negative (real) eigenvalues, so that $g(x) = \det g_{ij}(x)$, which is just the product of all eigenvalues, may be either positive or negative.

We consider an n-dimensional Riemannian (!) manifold M with coordinates x^i, $i = 1, \ldots, n = \dim \mathrm{M}$, and metric tensor $H(x) = \eta_{ij}(x)\, dx^i \otimes dx^j$, relative to a coordinate basis. We construct a symmetric rank-2 cotensor field $\Gamma : \mathrm{TM} \times \mathrm{TM} \to \mathbb{R}$ with the help of a covector field $\Omega \in \mathrm{T}^*\mathrm{M}$, with $\Omega(x) \in \mathrm{T}^*\mathrm{M}_x$:

$$\Omega(x) = \omega_i(x)dx^i\,.$$

Viz.

$$\Gamma(x) = \gamma_{ij}(x)\, dx^i \otimes dx^j \quad \text{with} \quad \gamma_{ij}(x) = \eta_{ij}(x) - \omega_i(x)\omega_j(x)\,.$$

The signature of a symmetric rank-2 cotensor is given by $(s_+(x), s_-(x), s_0(x))$, in which $s_\pm(x)$ is the dimension of the subspace spanned by all eigenvectors corresponding to positive/negative eigenvalues, and $s_0(x) = n - (s_+(x) + s_-(x))$, the dimension of the null space. If $s_0(x) > 0$ then $\Gamma(x)$ is called degenerate at $x \in \mathrm{M}$.

(10) **b.** Show that $\Gamma(x)$ is degenerate if and only if $\gamma(x) = \det \gamma_{ij}(x) = 0$.

Degeneracy $s_0(x) > 0$ is equivalent to the existence of a zero eigenvalue of $\gamma_{ij}(x)$, which in turn is equivalent to $\gamma(x) \stackrel{\text{def}}{=} \det \gamma_{ij}(x) = 0$.

We henceforth take $n = \dim \mathrm{M} = 2$.

(5) **c1.** Derive an implicit equation for the set $\Delta \subset \mathrm{M}$,

$$\Delta \stackrel{\text{def}}{=} \{x \in \mathrm{M} \mid \Gamma(x) \text{ is degenerate}\}$$

in terms of the covector field Ω and Riemannian metric H.

A direct computation reveals that $\gamma(x) = \eta(x)(1 - \eta^{ij}(x)\omega_i(x)\omega_j(x))$, with $\eta(x) \stackrel{\text{def}}{=} \det \eta_{ij}(x) > 0$. Thus degeneracy occurs at points $x \in$ M for which $\eta^{ij}(x)\omega_i(x)\omega_j(x) = 1$. In coordinate independent form one could write this as $\langle \Omega, H^*\Omega \rangle = 1$, or as $H^*(\Omega, \Omega) = 1$, in which H^* is the dual Riemannian metric field corresponding to H, and $\langle \_, \_ \rangle :$ T*M × TM → $\mathbb{R}$ the Kronecker tensor field.

(5) **c2.** Assuming $x \in \Delta$, provide an eigenvector $\mathbf{n}(x) = n^i(x)\partial_i \in \mathrm{TM}_x$ of $\Gamma(x)$ with a zero eigenvalue.

Take $n^i(x) = \eta^{ij}(x)\omega_j(x)$, then $\gamma_{ij}(x)n^j(x) = \gamma_{ij}(x)\eta^{jk}(x)\omega_k(x) = (\eta_{ij}(x) - \omega_i(x)\omega_j(x))\eta^{jk}(x)\omega_k(x) = \omega_i(x) - \omega_i(x)(\eta^{jk}(x)\omega_j(x)\omega_k(x))$. Since $x \in \Delta$ we observe that $\mathbf{n}(x) = H^*(x)\Omega(x)$, with components $n^i(x) = \eta^{jk}(x)\omega_k(x)$, has the desired property.

(**40**) **3.** FINSLER GEOMETRY.

Definition. Let M be an n-dimensional smooth manifold. A Finsler norm on M is defined as a function

$$F : \mathrm{TM} \to \mathbb{R}_0^+$$

with the following properties:

- F is smooth on $\mathrm{TM}\backslash\{0\} = \{(x, y) \in \mathrm{TM} \,|\, y \neq 0 \in \mathrm{TM}_x\}$;
- $F(x, y) > 0$ for all $(x, y) \in \mathrm{TM}\backslash\{0\}$;
- $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$;
- the so-called *Finsler metric* with holor $g_{ij}(x, y)$ is positive definite for all $(x, y) \in \mathrm{TM}\backslash\{0\}$:

$$g_{ij}(x, y) \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \quad (i, j = 1, \ldots, n)\,.$$

The value $F(x, y)$ is interpreted as the norm of vector $y \in \mathrm{TM}_x$ at base point $x \in \mathrm{M}$.

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree r if

$$f(\lambda y) = \lambda^r f(y) \quad \text{for all } \lambda > 0.$$

(10) **a.** Show that such an f satisfies the partial differential equation $y^i \dfrac{\partial f(y)}{\partial y^i} = r f(y)$.

Applying $d/d\lambda$ to the defining equation $f(\lambda y) = \lambda^r f(y)$, and subsequently setting $\lambda = 1$ produces the result.

b. Show that a Finsler norm has the following properties on $\mathrm{TM}\backslash\{0\}$:

($2\frac{1}{2}$) **b1.** $y^i \dfrac{\partial F(x, y)}{\partial y^i} = F(x, y)$.

$F(x, y)$ is homogeneous of degree $r = 1$ w.r.t. y, so this result follows directly from a (x is an irrelevant parameter).

($2\frac{1}{2}$) **b2.** $y^i \dfrac{\partial^2 F(x, y)}{\partial y^i \partial y^j} = 0$.

Differentiate the equation in b1 w.r.t. y^j: $0 = \dfrac{\partial}{\partial y^j}\left(y^i \dfrac{\partial F(x, y)}{\partial y^i} - F(x, y)\right) = \dfrac{\partial F(x, y)}{\partial y^j} + y^i \dfrac{\partial^2 F(x, y)}{\partial y^i \partial y^j} - \dfrac{\partial F(x, y)}{\partial y^j} = y^i \dfrac{\partial^2 F(x, y)}{\partial y^i \partial y^j}$.

($2\frac{1}{2}$) **b3.** $g_{ij}(x, y) = F(x, y)\dfrac{\partial^2 F(x, y)}{\partial y^i \partial y^j} + \dfrac{\partial F(x, y)}{\partial y^i}\dfrac{\partial F(x, y)}{\partial y^j}$.

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} = \frac{1}{2}\frac{\partial}{\partial y^i}\left(2F(x, y)\frac{\partial F(x, y)}{\partial y^j}\right) = F(x, y)\frac{\partial^2 F(x, y)}{\partial y^i \partial y^j} + \frac{\partial F(x, y)}{\partial y^i}\frac{\partial F(x, y)}{\partial y^j}.$$

($2\frac{1}{2}$) **b4.** $F^2(x, y) = g_{ij}(x, y)\, y^i y^j$.

Contract left and right hand sides of b3 with $y^i y^j$: $g_{ij}(x,y) y^i y^j = \left(F(x,y) \dfrac{\partial^2 F(x,y)}{\partial y^i \partial y^j} + \dfrac{\partial F(x,y)}{\partial y^i} \dfrac{\partial F(x,y)}{\partial y^j} \right) y^i y^j = F^2(x,y)$. The last step uses b1 and b2.

Consider the special case $F^2(x,y) \stackrel{\text{def}}{=} e^{2\psi(x,y)}\, a_{ij}(x)\, y^i y^j$, in which $\psi = C^\infty(\text{TM}\backslash\{0\})$.

(5) **c1.** Which homogeneity condition must ψ fulfill given that F defines a Finsler norm?

The function ψ must be homogeneous function of degree 0 in y, i.e. $\psi(x, \lambda y) = \psi(x,y)$.

(5) **c2.** Under which condition on ψ does F define a Riemannian (inner product) norm?

The function ψ must be independent of y, i.e. $\psi(x,y) = \psi(x)$, in which case the Finsler metric becomes a Riemannian metric, viz. $g_{ij}(x) = e^{2\psi(x)}\, a_{ij}(x)$. Only in this case the Finsler norm is the squareroot of a positive quadratic form.

(10) **d.** Assuming the condition under c1 to hold, derive the explicit formula for $g_{ij}(x,y)$.

A tedious but straightforward computation (suppressing the dependence on (x,y) for brevity) reveals that

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} = \dot{\partial}_{ij}\psi\, F^2 + 2\dot{\partial}_i \psi\, \dot{\partial}_j \psi + e^{2\psi} \left(a_{ij} + 2(\dot{\partial}_i \psi\, a_{jk} + \dot{\partial}_j \psi\, a_{ik}) y^k \right) .$$

The symbol $\dot{\partial}_i$ is shorthand for $\partial / \partial y^i$. The coordinate x is fixed throughout.