# EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY 

Course code: 2F800. Date: Thursday April 23, 2009. Time: 09h00-12h00. Place: HG 10.30 E.

## Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- You may consult the course notes "Tensorrekening en Differentiaalmeetkunde (2F800)" by Jan de Graaf, and the draft notes "Tensor Calculus and Differential Geometry (2F800)" by Luc Florack without warranty.


## 1. Hodge star on Lorentzian space.

Definition. A Lorentzian inner product on $n$-dimensional linear space $V$ is an indefinite symmetric bilinear mapping $(\cdot \mid \cdot): V \times V \rightarrow \mathbb{R}$ which satisfies the following axioms:

- $\forall \mathbf{x}, \mathbf{y} \in V: \quad(\mathbf{x} \mid \mathbf{y})=(\mathbf{y} \mid \mathbf{x})$,
- $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, \lambda, \mu \in \mathbb{R}: \quad(\lambda \mathbf{x}+\mu \mathbf{y} \mid \mathbf{z})=\lambda(\mathbf{x} \mid \mathbf{z})+\mu(\mathbf{y} \mid \mathbf{z})$,
- $\forall \mathbf{x} \in V$ with $\mathbf{x} \neq \mathbf{0} \exists \mathbf{y} \in V: \quad(\mathbf{x} \mid \mathbf{y}) \neq 0$.

Given a basis $\left\{\partial_{i}\right\}$ of $V$ and a corresponding decomposition $\mathbf{x}=x^{i} \partial_{i}, \mathbf{y}=y^{i} \partial_{i}$, we may write $(\mathbf{x} \mid \mathbf{y})=G(\mathbf{x}, \mathbf{y})=g_{i j} x^{i} y^{j}$, in which $g_{i j}=\left(\partial_{i} \mid \partial_{j}\right)$ are the components of the corresponding Gram matrix $\mathbf{G}$. The dual basis of $V^{*}$ is indicated by $\left\{d x^{i}\right\}$.

Definition. The converters $\sharp: V \rightarrow V^{*}$ and $b: V^{*} \rightarrow V$ are defined as follows. If $\mathbf{v} \in V, \hat{\mathbf{v}} \in V^{*}$, then

$$
\sharp \mathbf{v} \stackrel{\text { def }}{=} G(\mathbf{v}, \cdot) \in V^{*} \quad \text { respectively } \quad b \hat{\mathbf{v}} \stackrel{\text { def }}{=} G^{-1}(\hat{\mathbf{v}}, \cdot) \in V
$$

Definition. Let $\hat{\mathbf{a}}^{1}, \ldots, \hat{\mathbf{a}}^{k} \in V^{*}$. The Hodge star operator $*: \bigwedge_{k}(V) \rightarrow \bigwedge_{n-k}(V)$ is defined as follows:

$$
\left\{\begin{array}{lll}
* 1 & =\boldsymbol{\epsilon} & \text { for } k=0, \\
*\left(\hat{\mathbf{a}}^{1} \wedge \ldots \wedge \hat{\mathbf{a}}^{k}\right) & \left.\left.=\boldsymbol{\epsilon}\lrcorner b \hat{\mathbf{a}}^{1}\right\lrcorner \ldots\right\lrcorner b \hat{\mathbf{a}}^{k} & \text { for } k=1, \ldots, n .
\end{array}\right.
$$

In this definition $\boldsymbol{\epsilon}=\sqrt{|g|} d x^{1} \wedge \ldots \wedge d x^{n}=\epsilon_{\left|i_{1} \ldots i_{n}\right|} d x^{i_{1}} \wedge \ldots \wedge d x^{i_{n}} \in \bigwedge_{n}(V)$ is the Levi-Civita tensor, with $\epsilon_{1 \ldots n}=\sqrt{|g|}=\sqrt{|\operatorname{det} \mathbf{G}|}$.

Definition. Let $\mathbf{a}_{1}, \ldots, \mathbf{a}_{k} \in V$. The $(n-k)$-form $\left.\left.\left.\boldsymbol{\epsilon}\right\lrcorner \mathbf{a}_{1}\right\lrcorner \ldots\right\lrcorner \mathbf{a}_{k} \in \bigwedge_{n-k}(V)$ is defined as follows:

$$
\left.\left.\left.(\boldsymbol{\epsilon}\lrcorner \mathbf{a}_{1}\right\lrcorner \ldots\right\lrcorner \mathbf{a}_{k}\right)\left(\mathbf{x}_{k+1}, \ldots, \mathbf{x}_{n}\right)=\boldsymbol{\epsilon}\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{k}, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_{n}\right) \quad \text { for all } \mathbf{x}_{k+1}, \ldots, \mathbf{x}_{n} \in V
$$

In the following problems we take $n=3, V=\mathbb{R}^{3}$, and ("pseudo-Riemannian metric") $g_{11}=g_{22}=1$, $g_{33}=-1, g_{i j}=0$ otherwise $(i, j=1,2,3)$. Moreover, we make the following notational simplifications: $\partial_{1}=\partial_{x}, \partial_{2}=\partial_{y}, \partial_{3}=\partial_{t}, d x^{1}=d x, d x^{2}=d y, d x^{3}=d t$.
a. Let $\mathbf{x}=x^{i} \partial_{i} \in V, \mathbf{y}=y^{i} \partial_{i} \in V$. Show that $(\mathbf{x} \mid \mathbf{y})=g_{i j} x^{i} y^{j}$ defines a Lorentzian inner product.

Take $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\lambda, \mu \in \mathbb{R}$ arbitrary. Then

- $(\mathbf{x} \mid \mathbf{y})=g_{i j} x^{i} y^{j} \stackrel{*}{=} g_{j i} y^{j} x^{i}=(\mathbf{y} \mid \mathbf{x})$,
- $(\lambda \mathbf{x}+\mu \mathbf{y} \mid \mathbf{z})=g_{i j}\left(\lambda x^{i}+\mu y^{i}\right) z^{j}=\lambda g_{i j} x^{i} z^{j}+\mu g_{i j} y^{i} z^{j}=\lambda(\mathbf{x} \mid \mathbf{z})+\mu(\mathbf{y} \mid \mathbf{z})$,
- Suppose $x^{a}=\xi \neq 0$ for some $a=1,2,3$, take $y^{a}=\xi, y^{b}=0$ for $b \neq a$, then $(\mathbf{x} \mid \mathbf{y})= \pm \xi^{2} \neq 0$ (viz. + if $a=1,2,-$ if $a=3$ ).

Equality $*$ exploits the symmetry $g_{i j}=g_{j i}$.
b. Compute the following forms in terms of wedge products of $d x, d y$, and $d t$ :

-     * 1 ;
- $* d x, * d y, * d t$;
- $*(d x \wedge d y), *(d x \wedge d t), *(d y \wedge d t)$;
- $*(d x \wedge d y \wedge d t)$.

Note that $|g|=1$, thus $\epsilon_{123}=1$, whence $\epsilon_{i j k}=[i, j, k]$. Moreover, $b d x^{i}=g^{i j} \partial_{j}$, so that, using $g^{11}=g^{22}=-g^{33}=1$ we obtain $b d x=\partial_{x}, b d y=\partial_{y}, b d t=-\partial_{t}$. This is used in all of the computations below.

- *1= $\boldsymbol{\epsilon}=\epsilon_{|i j k|} d x^{i} \wedge d x^{j} \wedge d x^{k} \stackrel{*}{=} d x \wedge d y \wedge d t$.

In $*$ we have used the fact that there is only one effective term, viz. $(i, j, k)=(1,2,3)$. Furthermore,

- $* d x\left(\partial_{j}, \partial_{k}\right)=\boldsymbol{\epsilon}\left(b d x, \partial_{j}, \partial_{k}\right)=\boldsymbol{\epsilon}\left(\partial_{x}, \partial_{j}, \partial_{k}\right)=\epsilon_{1 j k}$. Therefore $* d x=\epsilon_{1|j k|} d x^{j} \wedge d x^{k} \stackrel{*}{=} d y \wedge d t$.
- $* d y\left(\partial_{j}, \partial_{k}\right)=\boldsymbol{\epsilon}\left(b d y, \partial_{j}, \partial_{k}\right)=\boldsymbol{\epsilon}\left(\partial_{y}, \partial_{j}, \partial_{k}\right)=\epsilon_{2 j k}$. Therefore $* d y=\epsilon_{2|j k|} d x^{j} \wedge d x^{k} \stackrel{\star}{=}-d x \wedge d t$.
- $* d t\left(\partial_{j}, \partial_{k}\right)=\boldsymbol{\epsilon}\left(b d t, \partial_{j}, \partial_{k}\right)=\boldsymbol{\epsilon}\left(-\partial_{t}, \partial_{j}, \partial_{k}\right)=-\epsilon_{3 j k}$. Therefore $* d t=-\epsilon_{3|j k|} d x^{j} \wedge d x^{k} \stackrel{\circ}{\doteq}-d x \wedge d y$.

In $*$ we have used the fact that there is only one effective term, viz. $(j, k)=(2,3)$. In $\star$ we have used the fact that there is only one effective term, viz. $(j, k)=(1,3)$. In o we have used the fact that there is only one effective term, viz. $(j, k)=(1,2)$.

- $*(d x \wedge d y)\left(\partial_{k}\right)=\boldsymbol{\epsilon}\left(b d x, b d y, \partial_{k}\right)=\boldsymbol{\epsilon}\left(\partial_{x}, \partial_{y}, \partial_{k}\right)=\epsilon_{12 k}$. Therefore $*(d x \wedge d y)=\epsilon_{12 k} d x^{k} \stackrel{*}{=} d t$.
- $*(d x \wedge d t)\left(\partial_{k}\right)=\boldsymbol{\epsilon}\left(b d x, b d t, \partial_{k}\right)=\boldsymbol{\epsilon}\left(\partial_{x},-\partial_{t}, \partial_{k}\right)=-\epsilon_{13 k}$. Therefore $*(d x \wedge d t)=-\epsilon_{13 k} d x^{k} \stackrel{\star}{=} d y$.
- $*(d y \wedge d t)\left(\partial_{k}\right)=\boldsymbol{\epsilon}\left(b d y, b d t, \partial_{k}\right)=\boldsymbol{\epsilon}\left(\partial_{y},-\partial_{t}, \partial_{k}\right)=-\epsilon_{23 k}$. Therefore $*(d y \wedge d t)=-\epsilon_{23 k} d x^{k} \stackrel{\circ}{=}-d x$.

In $*$ we have used the fact that there is only one effective term, viz. $k=3$. In $\star$ we have used the fact that there is only one effective term, viz. $k=2$. In $\circ$ we have used the fact that there is only one effective term, viz. $k=1$. Finally,

- $*(d x \wedge d y \wedge d t)=\boldsymbol{\epsilon}(b d x, b d y, b d t)=\boldsymbol{\epsilon}\left(\partial_{x}, \partial_{y},-\partial_{t}\right)=-\epsilon_{123}=-1$.
(5) c. What can you say about the double Hodge star operator $* *: \bigwedge_{k}(V) \rightarrow \bigwedge_{k}(V)(k=0,1,2,3)$ ?

By inspection of the previous results under be obtain, respectively,

- $* * 1=* \boldsymbol{\epsilon}=*(d x \wedge d y \wedge d t)=-1$.
- $* * d x=*(d y \wedge d t)=-d x$.
- $* * d y=-*(d x \wedge d t)=-d y$.
- **dt $=-*(d x \wedge d y)=-d t$.
- $* *(d x \wedge d y)=* d t=-d x \wedge d y$.
- ** $(d x \wedge d t)=* d y=-d x \wedge d t$.
- ** $(d y \wedge d t)=-* d x=-d y \wedge d t$.
- ** $(d x \wedge d y \wedge d t)=-* 1=-d x \wedge d y \wedge d t$.

All in all we observe that $* *=-\mathrm{id} \wedge_{k}(V)$ for any $k=0,1,2,3$, so formally $* *=-1$. (In general $* *=(-1)^{k(n-k)} \operatorname{sgn} g \operatorname{id}_{\wedge_{k}(V)}$.)

## (20) 2. ANTISYMMETRISATION.

Notation. By $\pi=\left\{\pi\left(i_{1}\right), \ldots, \pi\left(i_{p}\right)\right\}$ we denote a permutation of the $p$ ordered symbols $\left\{i_{1}, \ldots, i_{p}\right\}$. The notation $\pi\left(i_{k}\right), k=1, \ldots, p$, is shorthand for the $k$-th entry of $\pi$. By $\operatorname{sgn}(\pi)= \pm 1$ we indicate the parity (even/odd) of $\pi$. Furthermore, $V$ is an $n$-dimensional linear space with basis $\left\{\mathbf{e}_{i}\right\}$. The corresponding basis of $V^{*}$ is indicated by $\left\{\hat{\mathbf{e}}^{i}\right\}$.

Definition. The antisymmetrisation map $\mathscr{A}: \mathbf{T}_{p}^{0}(V) \longrightarrow \bigwedge_{p}(V)$ is defined as follows:

$$
(\mathscr{A}(\mathbf{T}))\left(\mathbf{v}_{1}, \ldots, \mathbf{v}_{p}\right)=\frac{1}{p!} \sum_{\pi} \operatorname{sgn}(\pi) \mathbf{T}\left(\mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(p)}\right)
$$

Let $\mathbf{T}=t_{i_{1} \ldots i_{p}} \hat{\mathbf{e}}^{i_{1}} \otimes \ldots \otimes \hat{\mathbf{e}}^{i_{p}} \in \mathbf{T}_{p}^{0}(V)$. Show that

$$
\mathscr{A}(\mathbf{T})=t_{\left[i_{1} \ldots i_{p}\right]} \hat{\mathbf{e}}^{i_{1}} \otimes \ldots \otimes \hat{\mathbf{e}}^{i_{p}}
$$

in which the antisymmetrised holor is defined as follows:

$$
t_{\left[i_{1} \ldots i_{p}\right]} \stackrel{\text { def }}{=} \frac{1}{p!} \sum_{\pi} \operatorname{sgn}(\pi) t_{\pi\left(i_{1}\right) \ldots \pi\left(i_{p}\right)}
$$

The holor is obtained by evaluating the tensor on (any combination of) basis vectors:

$$
(\mathscr{A}(\mathbf{T}))\left(\mathbf{e}_{i_{1}}, \ldots, \mathbf{e}_{i_{p}}\right)=\frac{1}{p!} \sum_{\pi} \operatorname{sgn}(\pi) \mathbf{T}\left(\mathbf{e}_{\pi\left(i_{1}\right)}, \ldots, \mathbf{e}_{\pi\left(i_{p}\right)}\right)=\frac{1}{p!} \sum_{\pi} \operatorname{sgn}(\pi) t_{\pi\left(i_{1}\right) \ldots \pi\left(i_{p}\right)}=t_{\left[i_{1} \ldots i_{p}\right]} .
$$

3. Relative tensors.

Definition. Let A be a square matrix with entries $A_{i j}(i, j=1, \ldots, n)$. The cofactor matrix of $\mathbf{A}$ is the matrix $\tilde{\mathbf{A}}$ with entries $\tilde{A}^{i j}$ given by

$$
\tilde{A}^{i j}=\frac{\partial \operatorname{det} \mathbf{A}}{\partial A_{i j}} .
$$

a. Compute the cofactor matrix $\tilde{\mathbf{A}}$ of

$$
\mathbf{A}=\left(\begin{array}{lll}
A_{11} & A_{12} & A_{13}  \tag{5}\\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{array}\right)
$$

(Hint: $\operatorname{det} A=A_{11} A_{22} A_{33}+A_{12} A_{23} A_{31}+A_{13} A_{21} A_{32}-A_{13} A_{22} A_{31}-A_{11} A_{23} A_{32}-A_{12} A_{21} A_{33}$.)
We have

$$
\operatorname{det} A=A_{11} A_{22} A_{33}+A_{12} A_{23} A_{31}+A_{13} A_{21} A_{32}-A_{13} A_{22} A_{31}-A_{11} A_{23} A_{32}-A_{12} A_{21} A_{33},
$$

whence

$$
\tilde{\mathbf{A}}=\left(\begin{array}{lll}
A_{22} A_{33}-A_{23} A_{32} & A_{23} A_{31}-A_{12} A_{33} & A_{21} A_{32}-A_{22} A_{31} \\
A_{13} A_{32}-A_{12} A_{33} & A_{11} A_{33}-A_{13} A_{31} & A_{12} A_{31}-A_{11} A_{32} \\
A_{12} A_{23}-A_{13} A_{22} & A_{13} A_{21}-A_{11} A_{23} & A_{11} A_{22}-A_{12} A_{21}
\end{array}\right) .
$$

Lemma. Let $T \in \mathbf{T}_{q}^{p}(V)$ be a mixed tensor, $\left\{\mathbf{e}_{i}\right\}$ a basis of the $n$-dimensional real linear space $V$, $\mathbf{f}_{j}=A_{j}^{i} \mathbf{e}_{i}$ a change of basis, with transformation matrix $A$, and $B=A^{-1}$, i.e. $A_{k}^{i} B_{j}^{k}=\delta_{j}^{i}$. Then

$$
T=T_{j_{1} \ldots j_{q}}^{i_{1} \ldots i_{p}} \mathbf{e}_{i_{1}} \otimes \ldots \otimes \mathbf{e}_{i_{p}} \otimes \hat{\mathbf{e}}^{j_{1}} \otimes \ldots \otimes \hat{\mathbf{e}}^{j_{q}} \stackrel{\text { def }}{=} \bar{T}_{j_{1} \ldots j_{q}}^{i_{1} \ldots i_{p}} \mathbf{f}_{i_{1}} \otimes \ldots \otimes \mathbf{f}_{i_{p}} \otimes \hat{\mathbf{f}}^{j_{1}} \otimes \ldots \otimes \hat{\mathbf{f}}^{j_{q}}
$$

if and only if the holor adheres to the tensor transformation law,

$$
\bar{T}_{j_{1} \ldots j_{q}}^{i_{1} \ldots i_{p}}=A_{j_{1}}^{\ell_{1}} \ldots A_{j_{q}}^{\ell_{q}} B_{k_{1}}^{i_{1}} \ldots B_{k_{p}}^{i_{p}} T_{\ell_{1} \ldots \ell_{q}}^{k_{1} \ldots k_{p}} .
$$

A scalar is defined as a tensor of type $(p, q)=(0,0)$.
In the following problems we take $p=0, q=2$. For notational simplicity we write $\mathbf{S}$ instead of $\overline{\mathbf{T}}$, and likewise $S_{i j}$ instead of $\bar{T}_{i j}$, for the corresponding matrix and holor representations after transformation. The tensor is invariably denoted by the symbol $T \in \mathbf{T}_{2}^{0}(V)$. In particular,

$$
T=T_{i j} \hat{\mathbf{e}}^{i} \otimes \hat{\mathbf{e}}^{j}=S_{i j} \hat{\mathbf{f}}^{i} \otimes \hat{\mathbf{f}}^{j} .
$$

b. Show that $T_{k \ell}=B_{k}^{i} B_{\ell}^{j} S_{i j}$.

The tensor transformation law states that $S_{i j}=A_{i}^{k} A_{j}^{\ell} T_{k \ell}$. Multiplication on both sides with $B_{p}^{i} B_{q}^{j}$, using $B_{p}^{i} A_{i}^{k}=\delta_{p}^{k}$ and $B_{q}^{j} A_{j}^{\ell}=\delta_{q}^{\ell}$, yields the desired result. Or, more directly, plug in the covector basis transformation $\hat{\mathbf{f}}^{i}=B_{p}^{i} \hat{\mathbf{e}}^{p}$ and $\hat{\mathbf{f}}^{j}=B_{q}^{j} \hat{\mathbf{e}}^{q}$ into $T=S_{i j} \hat{\mathbf{f}}^{i} \otimes \hat{\mathbf{f}}^{j}$ and read off the coefficients $T_{p q}$ in terms of $S_{i j}$ relative to the old basis $\left\{\mathbf{e}_{i}\right\}$.
c. Show that $\operatorname{det} \mathbf{T}$ is not a scalar by showing that $\operatorname{det} \mathbf{S}=(\operatorname{det} \mathbf{A})^{2} \operatorname{det} \mathbf{T}$.

From $S_{i j}=A_{i}^{k} A_{j}^{\ell} T_{k \ell}$ it follows immediately that $\operatorname{det} \mathbf{S}=(\operatorname{det} \mathbf{A})^{2} \operatorname{det} \mathbf{T} \neq \operatorname{det} \mathbf{T}$ (generically).
(5) d. Show that, consequently, the collection of numbers $\tilde{S}^{i j}$ does not transform as the holor of a rank-2 contratensor, and derive the correct "relative tensor transformation law" for it.
$\tilde{S}^{i j} \stackrel{\text { def }}{=} \partial \operatorname{det} \mathbf{S} / \partial S_{i j} \stackrel{*}{=} \partial T_{k \ell} / \partial S_{i j} \partial\left((\operatorname{det} \mathbf{A})^{2} \operatorname{det} \mathbf{T}\right) / \partial T_{k \ell} \stackrel{\star}{=}(\operatorname{det} \mathbf{A})^{2} B_{k}^{i} B_{\ell}^{j} \partial \operatorname{det} \mathbf{T} / \partial T_{k \ell} \stackrel{\operatorname{def}}{=}(\operatorname{det} \mathbf{A})^{2} B_{k}^{i} B_{\ell}^{j} \tilde{T}^{k \ell}$. The equality marked by $*$ exploits item $c$ as well as the chain rule, the one marked by $\star$ is based on $b$, the rest follows by definition of the cofactor matrix. The factor $(\operatorname{det} \mathbf{A})^{2}$ on the r.h.s. violates the usual tensor transformation law for a rank- 2 contratensor.

Below we assume that $V$ is equipped with a real inner product, with $(\mathbf{x} \mid \mathbf{y})=G(\mathbf{x}, \mathbf{y})=g_{i j} x^{i} y^{j}$, in which $g_{i j}=\left(\mathbf{e}_{i} \mid \mathbf{e}_{j}\right)$ are the corresponding components of the Gram matrix $\mathbf{G}$ relative to basis $\left\{\mathbf{e}_{i}\right\}$, with $g=\operatorname{det} \mathbf{G}$.
(5) e. We stipulate that $V^{i j}=g^{w} \tilde{T}^{i j}$, for some $w \in \mathbb{R}$, is the holor of a rank-2 contratensor that does adhere to the usual tensor transformation law. Prove that such a $w \in \mathbb{R}$ exists and determine its value.

Let $\bar{V}^{i j}=\bar{g}^{w} \tilde{S}^{i j}$. Substitution of $\bar{g} \stackrel{\text { c }}{=}(\operatorname{det} \mathbf{A})^{2} g$ and $\tilde{S}^{i j} \stackrel{\text { d }}{=}(\operatorname{det} \mathbf{A})^{2} B_{k}^{i} B_{\ell}^{j} \tilde{T}^{k \ell}$ yields $\bar{V}^{i j}=(\operatorname{det} \mathbf{A})^{2 w} g^{w}(\operatorname{det} \mathbf{A})^{2} B_{k}^{i} B_{\ell}^{j} \tilde{T}^{k \ell}=$ $B_{k}^{i} B_{\ell}^{j} V^{k \ell}$ if and only if $w=-1$.
(25) 4. Log-Polar retina.

We consider a simplified model of the human retina in the form of a 2 -dimensional manifold with a basic volume form $\boldsymbol{\mu} \in \bigwedge_{2}\left(\mathbb{R}^{2}\right)$ given in polar coordinates by

$$
\boldsymbol{\mu}=\frac{1}{r} d r \wedge d \phi
$$

Recall the definition of polar coordinates $(r, \phi)$ in terms of Cartesian coordinates $(x, y)$ :

$$
\left\{\begin{array}{l}
x=r \cos \phi \\
y=r \sin \phi
\end{array}\right.
$$

(We ignore subtleties that normally need to be considered in the context of reparametrizations, such as injectivity, differentiability and invertibility.)


Mapping (Connolly and Van Essen 1984) of the Visual field (A) on the LGN (B) AND THE STRIATE CORTEX (C) IN MONKEY. THE REPRESENTATION OF THE CENTRAL 5 DEGREES (SHADED AREAS) IN THE VISUAL FIELD OCCUPIES ABOUT $40 \%$ OF THE CORTEX. THE VOLUME FORM $\boldsymbol{\mu}$ IN THIS PROBLEM ANATOMICALLY REPRESENTS A DIFFERENTIAL SURFACE ELEMENT ON THE HUMAN STRIATE CORTEX RETINOTOPICALLY CONNECTED TO A CONSTANT SURFACE ELEMENT $d x \wedge d y$ ON THE RETINA AT POSITION $(x, y)$ RELATIVE TO THE FOVEAL CENTER. PERCEPTUALLY IT EXPLAINS THE LINEAR DECREASE OF RESOLVING POWER AS A FUNCTION OF ECCENTRICITY $r$.
a1. Express the 2 -form $\boldsymbol{\mu}$ in $(x, y)$ coordinates.

From $r=\sqrt{x^{2}+y^{2}}$ and $\phi=\arctan \frac{y}{x}$ it follows that $d r=\frac{x d x+y d y}{\sqrt{x^{2}+y^{2}}}$ and $d \phi=\frac{x d y-y d x}{x^{2}+y^{2}}$, whence, using nilpotency and (thus) antisymmetry of the wedge product $(d x \wedge d x=d y \wedge d y=0, d x \wedge d y=-d y \wedge d x), \boldsymbol{\mu}=\frac{1}{r} d r \wedge d \phi=\frac{1}{x^{2}+y^{2}} d x \wedge d y$.
a2. Find "log-polar coordinates" coordinates $(\rho, \psi)$ such that $\boldsymbol{\mu}=d \rho \wedge d \psi$.
The symmetry of the problem suggests a transformation of the form

$$
\left\{\begin{array}{l}
\rho=f(r) \\
\psi=\phi
\end{array}\right.
$$

This implies $d \rho=f^{\prime}(r) d r$ and $d \psi=d \phi$, whence $\boldsymbol{\mu}=d \rho \wedge d \psi=f^{\prime}(r) d r \wedge d \phi$. We conclude that $f^{\prime}(r)=1 / r$, so that we may take $f(r)=\ln r$ (up to an additive constant). (This is also immediately evident from the fact that $d r / r=d \ln r$.)

Definition. The Christoffel symbols $\Gamma_{i j}^{k}$ associated with the affine connection $\nabla: \mathrm{TR}^{2} \times \mathrm{TR}^{2} \rightarrow \mathrm{TR}^{2}$
are defined by $\nabla_{\partial_{j}} \partial_{i}=\Gamma_{i j}^{k} \partial_{k}(i=1,2)$.
b. Show that the components of the Christoffel symbols in a coordinate system $\overline{\mathrm{x}}$ are given relative to those in a coordinate system x by

$$
\bar{\Gamma}_{i j}^{k}=S_{\ell}^{k}\left(T_{j}^{m} T_{i}^{n} \Gamma_{n m}^{\ell}+\bar{\partial}_{j} T_{i}^{\ell}\right)
$$

in which $\mathbf{T}$ and $\mathbf{S}$ are the Jacobian matrices with components

$$
T_{j}^{i}=\frac{\partial x^{i}}{\partial \bar{x}^{j}} \quad \text { resp. } \quad S_{j}^{i}=\frac{\partial \bar{x}^{i}}{\partial x^{j}} .
$$

We have

$$
\bar{\Gamma}_{i j}^{k}=\left\langle d \bar{x}^{k}, \nabla_{\bar{\partial}_{j}} \bar{\partial}_{i}\right\rangle .
$$

Substitute the transformations, and note that $\nabla_{\bar{\partial}_{j}}=T_{j}^{\ell} \nabla_{\partial_{\ell}}$, respectively $\nabla_{\partial_{\ell}}\left(T_{i}^{\ell} \partial_{k}\right)=\partial_{\ell} T_{i}^{\ell} \partial_{k}+T_{i}^{\ell} \nabla_{\partial_{\ell}} \partial_{k}$. Subsequently use the definition of the Christoffel symbols, $\nabla_{\partial_{\ell}} \partial_{k}=\Gamma_{k \ell}^{m} \partial_{m}$, and finally the chain rule, $\bar{\partial}_{j}=T_{j}^{\ell} \partial_{\ell}$.
c. Under the assumption that $\Gamma_{i j}^{k}=0$ in Cartesian coordinates $(x, y)$, evaluate the induced symbols $\bar{\Gamma}_{i j}^{k}$ in log-polar coordinates $(\rho, \psi)$.

We now have

$$
\bar{\Gamma}_{i j}^{k}=S_{\ell}^{k} \bar{\partial}_{j} T_{i}^{\ell}
$$

Furthermore,

$$
\left\{\begin{array}{l}
x=e^{\rho} \cos \psi \\
y=e^{\rho} \sin \psi
\end{array}\right.
$$

If we interpret $\bar{\Gamma}_{i j}^{k}$ as a matrix with row index $k$ and column index $i$ for fixed $j, S_{\ell}^{k}$ as a matrix with row index $k$ and column index $\ell$, and $T_{i}^{\ell}$ as a matrix with row index $\ell$ and column index $i$, then (writing $\rho$ and $\psi$ for index values 1 and 2 , respectively)

$$
\mathbf{S}=\left(\begin{array}{cc}
e^{-\rho} \cos \psi & e^{-\rho} \sin \psi \\
-e^{-\rho} \sin \psi & e^{-\rho} \cos \psi
\end{array}\right) \quad \text { resp. } \quad \mathbf{T}=\left(\begin{array}{cc}
e^{\rho} \cos \psi & -e^{\rho} \sin \psi \\
e^{\rho} \sin \psi & e^{\rho} \cos \psi
\end{array}\right)
$$

so that

$$
\bar{\Gamma}_{i \rho}^{k}=\left(\begin{array}{cc}
e^{-\rho} \cos \psi & e^{-\rho} \sin \psi \\
-e^{-\rho} \sin \psi & e^{-\rho} \cos \psi
\end{array}\right) \partial_{\rho}\left(\begin{array}{cc}
e^{\rho} \cos \psi & -e^{\rho} \sin \psi \\
e^{\rho} \sin \psi & e^{\rho} \cos \psi
\end{array}\right)=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)
$$

and

$$
\bar{\Gamma}_{i \psi}^{k}=\left(\begin{array}{cc}
e^{-\rho} \cos \psi & e^{-\rho} \sin \psi \\
-e^{-\rho} \sin \psi & e^{-\rho} \cos \psi
\end{array}\right) \partial_{\psi}\left(\begin{array}{cc}
e^{\rho} \cos \psi & -e^{\rho} \sin \psi \\
e^{\rho} \sin \psi & e^{\rho} \cos \psi
\end{array}\right)=\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)
$$

This establishes all Christoffel symbols in log-polar coordinates: $\bar{\Gamma}_{\rho \rho}^{\rho}=1, \bar{\Gamma}_{\psi \rho}^{\psi}=1, \bar{\Gamma}_{\psi \psi}^{\rho}=-1, \bar{\Gamma}_{\rho \psi}^{\psi}=1$, all other symbols zero.
(5) d. As opposed to the previous problem we now assume that $\bar{\Gamma}_{i j}^{k}=0$ in log-polar coordinates $(\rho, \psi)$. Evaluate the induced symbols $\Gamma_{i j}^{k}$ in Cartesian coordinates $(x, y)$.

Interchanging the roles of the matrices $\mathbf{S}$ and $\mathbf{T}$, those of $\partial_{j}$ and $\bar{\partial}_{j}$, as well as those of $\Gamma_{i j}^{k}$ and $\bar{\Gamma}_{i j}^{k}$ in the transformation formula for the Christoffel symbols, recall b, yields

$$
\Gamma_{i j}^{k}=T_{\ell}^{k}\left(S_{j}^{m} S_{i}^{n} \bar{\Gamma}_{n m}^{\ell}+\partial_{j} S_{i}^{\ell}\right)
$$

In particular, in the case at hand,

$$
\Gamma_{i j}^{k}=T_{\ell}^{k} \partial_{j} S_{i}^{\ell}
$$

The matrices have been computed above, but need to be re-expressed in $(x, y)$ coordinates. In matrix notation, as before, we now have

$$
\Gamma_{i x}^{k}=\left(\begin{array}{cc}
x & -y \\
y & x
\end{array}\right) \partial_{x}\left(\begin{array}{cc}
\frac{x}{x^{2}+y^{2}} & \frac{y}{x^{2}+y^{2}} \\
-\frac{x}{x^{2}+y^{2}} & \frac{x}{x^{2}+y^{2}}
\end{array}\right)=\left(\begin{array}{cc}
-\frac{x}{x^{2}+y^{2}} & -\frac{y}{x^{2}+y^{2}} \\
\frac{y}{x^{2}+y^{2}} & -\frac{x}{x^{2}+y^{2}}
\end{array}\right)
$$

and

$$
\Gamma_{i y}^{k}=\left(\begin{array}{cc}
x & -y \\
y & x
\end{array}\right) \partial_{y}\left(\begin{array}{cc}
\frac{x}{x^{2}+y^{2}} & \frac{y}{x^{2}+y^{2}} \\
-\frac{x}{x^{2}+y^{2}} & \frac{x}{x^{2}+y^{2}}
\end{array}\right)=\left(\begin{array}{cc}
-\frac{y}{x^{2}+y^{2}} & \frac{x}{x^{2}+y^{2}} \\
-\frac{x}{x^{2}+y^{2}} & -\frac{x^{2}+y^{2}}{2}
\end{array}\right)
$$

That is, $\Gamma_{x x}^{x}=\Gamma_{y x}^{y}=\Gamma_{x y}^{y}=-\Gamma_{y y}^{x}=-x /\left(x^{2}+y^{2}\right)$ and $\Gamma_{y x}^{x}=-\Gamma_{x x}^{y}=\Gamma_{x y}^{x}=\Gamma_{y y}^{y}=-y /\left(x^{2}+y^{2}\right)$.

## THE END

