# EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2F800. Date: Thursday April 23, 2009. Time: 09h00–12h00. Place: HG 10.30 E.

**Read this first!**

- Write your name and student identification number on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- You may consult the course notes "Tensorrekening en Differentiaalmeetkunde (2F800)" by Jan de Graaf, and the draft notes "Tensor Calculus and Differential Geometry (2F800)" by Luc Florack without warranty.

(**30**) **1.** HODGE STAR ON LORENTZIAN SPACE.

**Definition.** A Lorentzian inner product on $n$-dimensional linear space $V$ is an indefinite symmetric bilinear mapping $(\,\cdot\,|\,\cdot\,) : V \times V \to \mathbb{R}$ which satisfies the following axioms:

- $\forall\, \mathbf{x}, \mathbf{y} \in V : \quad (\mathbf{x}|\mathbf{y}) = (\mathbf{y}|\mathbf{x})$,
- $\forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, \lambda, \mu \in \mathbb{R} : \quad (\lambda\mathbf{x} + \mu\mathbf{y}|\mathbf{z}) = \lambda\,(\mathbf{x}|\mathbf{z}) + \mu\,(\mathbf{y}|\mathbf{z})$,
- $\forall\, \mathbf{x} \in V$ with $\mathbf{x} \neq \mathbf{0}\ \exists\, \mathbf{y} \in V : \quad (\mathbf{x}|\mathbf{y}) \neq 0$.

Given a basis $\{\partial_i\}$ of $V$ and a corresponding decomposition $\mathbf{x} = x^i\partial_i$, $\mathbf{y} = y^i\partial_i$, we may write $(\mathbf{x}|\mathbf{y}) = G(\mathbf{x},\mathbf{y}) = g_{ij}x^iy^j$, in which $g_{ij} = (\partial_i|\partial_j)$ are the components of the corresponding Gram matrix $\mathbf{G}$. The dual basis of $V^*$ is indicated by $\{dx^i\}$.

**Definition.** The converters $\sharp : V \to V^*$ and $\flat : V^* \to V$ are defined as follows. If $\mathbf{v} \in V$, $\hat{\mathbf{v}} \in V^*$, then

$$\sharp\mathbf{v} \stackrel{\text{def}}{=} G(\mathbf{v},\,\cdot\,) \in V^* \quad \text{respectively} \quad \flat\hat{\mathbf{v}} \stackrel{\text{def}}{=} G^{-1}(\hat{\mathbf{v}},\,\cdot\,) \in V\,.$$

**Definition.** Let $\hat{\mathbf{a}}^1, \ldots, \hat{\mathbf{a}}^k \in V^*$. The Hodge star operator $* : \bigwedge_k(V) \to \bigwedge_{n-k}(V)$ is defined as follows:

$$\begin{cases} *1 & = \quad \boldsymbol{\epsilon} & \text{for } k = 0, \\ *\left(\hat{\mathbf{a}}^1 \wedge \ldots \wedge \hat{\mathbf{a}}^k\right) & = \quad \boldsymbol{\epsilon}\lrcorner\flat\hat{\mathbf{a}}^1\lrcorner\ldots\lrcorner\flat\hat{\mathbf{a}}^k & \text{for } k = 1, \ldots, n. \end{cases}$$

In this definition $\boldsymbol{\epsilon} = \sqrt{|g|}\, dx^1 \wedge \ldots \wedge dx^n = \epsilon_{|i_1\ldots i_n|}dx^{i_1} \wedge \ldots \wedge dx^{i_n} \in \bigwedge_n(V)$ is the Levi-Civita tensor, with $\epsilon_{1\ldots n} = \sqrt{|g|} = \sqrt{|\det \mathbf{G}|}$.

**Definition.** Let $\mathbf{a}_1, \ldots, \mathbf{a}_k \in V$. The $(n-k)$-form $\boldsymbol{\epsilon}\lrcorner\mathbf{a}_1\lrcorner\ldots\lrcorner\mathbf{a}_k \in \bigwedge_{n-k}(V)$ is defined as follows:

$$\left(\boldsymbol{\epsilon}\lrcorner\mathbf{a}_1\lrcorner\ldots\lrcorner\mathbf{a}_k\right)(\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n) = \boldsymbol{\epsilon}\,(\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n) \quad \text{for all } \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n \in V.$$

In the following problems we take $n = 3$, $V = \mathbb{R}^3$, and ("pseudo-Riemannian metric") $g_{11} = g_{22} = 1$, $g_{33} = -1$, $g_{ij} = 0$ otherwise ($i, j = 1, 2, 3$). Moreover, we make the following notational simplifications: $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\partial_3 = \partial_t$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dt$.

(5) **a.** Let $\mathbf{x} = x^i \partial_i \in V$, $\mathbf{y} = y^i \partial_i \in V$. Show that $(\mathbf{x}|\mathbf{y}) = g_{ij} x^i y^j$ defines a Lorentzian inner product.

(20) **b.** Compute the following forms in terms of wedge products of $dx$, $dy$, and $dt$:

- $*\,1$;
- $*\,dx$, $*\,dy$, $*\,dt$;
- $*\,(dx \wedge dy)$, $*\,(dx \wedge dt)$, $*\,(dy \wedge dt)$;
- $*\,(dx \wedge dy \wedge dt)$.

(5) **c.** What can you say about the double Hodge star operator $** : \bigwedge_k(V) \to \bigwedge_k(V)$ ($k = 0, 1, 2, 3$)?

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(**20**) **2.** ANTISYMMETRISATION.

**Notation.** By $\pi = \{\pi(i_1), \ldots, \pi(i_p)\}$ we denote a permutation of the $p$ ordered symbols $\{i_1, \ldots, i_p\}$. The notation $\pi(i_k)$, $k = 1, \ldots, p$, is shorthand for the $k$-th entry of $\pi$. By $\operatorname{sgn}(\pi) = \pm 1$ we indicate the parity (even/odd) of $\pi$. Furthermore, $V$ is an $n$-dimensional linear space with basis $\{\mathbf{e}_i\}$. The corresponding basis of $V^*$ is indicated by $\{\hat{\mathbf{e}}^i\}$.

**Definition.** The antisymmetrisation map $\mathscr{A} : \mathbf{T}^0_p(V) \longrightarrow \bigwedge_p(V)$ is defined as follows:

$$(\mathscr{A}(\mathbf{T}))\,(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \frac{1}{p!} \sum_\pi \operatorname{sgn}(\pi) \mathbf{T}(\mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(p)})\,.$$

Let $\mathbf{T} = t_{i_1 \ldots i_p} \hat{\mathbf{e}}^{i_1} \otimes \ldots \otimes \hat{\mathbf{e}}^{i_p} \in \mathbf{T}^0_p(V)$. Show that

$$\mathscr{A}(\mathbf{T}) = t_{[i_1 \ldots i_p]} \hat{\mathbf{e}}^{i_1} \otimes \ldots \otimes \hat{\mathbf{e}}^{i_p}\,,$$

in which the antisymmetrised holor is defined as follows:

$$t_{[i_1 \ldots i_p]} \stackrel{\text{def}}{=} \frac{1}{p!} \sum_\pi \operatorname{sgn}(\pi) t_{\pi(i_1) \ldots \pi(i_p)}\,.$$

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(**25**) **3.** RELATIVE TENSORS.

**Definition.** Let $\mathbf{A}$ be a square matrix with entries $A_{ij}$ $(i,j=1,\ldots,n)$. The cofactor matrix of $\mathbf{A}$ is the matrix $\tilde{\mathbf{A}}$ with entries $\tilde{A}^{ij}$ given by

$$\tilde{A}^{ij} = \frac{\partial \det \mathbf{A}}{\partial A_{ij}} \,.$$

(5) **a.** Compute the cofactor matrix $\tilde{\mathbf{A}}$ of

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} .$$

(*Hint:* $\det A = A_{11}A_{22}A_{33}+A_{12}A_{23}A_{31}+A_{13}A_{21}A_{32}-A_{13}A_{22}A_{31}-A_{11}A_{23}A_{32}-A_{12}A_{21}A_{33}$.)

**Lemma.** Let $T \in \mathbf{T}^p_q(V)$ be a mixed tensor, $\{\mathbf{e}_i\}$ a basis of the $n$-dimensional real linear space $V$, $\mathbf{f}_j = A^i_j \mathbf{e}_i$ a change of basis, with transformation matrix $A$, and $B = A^{-1}$, i.e. $A^i_k B^k_j = \delta^i_j$. Then

$$T = T^{i_1\ldots i_p}_{j_1\ldots j_q} \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_p} \otimes \hat{\mathbf{e}}^{j_1} \otimes \ldots \otimes \hat{\mathbf{e}}^{j_q} \stackrel{\text{def}}{=} \overline{T}^{i_1\ldots i_p}_{j_1\ldots j_q} \mathbf{f}_{i_1} \otimes \ldots \otimes \mathbf{f}_{i_p} \otimes \hat{\mathbf{f}}^{j_1} \otimes \ldots \otimes \hat{\mathbf{f}}^{j_q} \,,$$

if and only if the holor adheres to the tensor transformation law,

$$\overline{T}^{i_1\ldots i_p}_{j_1\ldots j_q} = A^{\ell_1}_{j_1} \ldots A^{\ell_q}_{j_q} B^{i_1}_{k_1} \ldots B^{i_p}_{k_p} T^{k_1\ldots k_p}_{\ell_1\ldots \ell_q} \,.$$

A scalar is defined as a tensor of type $(p,q)=(0,0)$.

In the following problems we take $p=0$, $q=2$. For notational simplicity we write $\mathbf{S}$ instead of $\overline{\mathbf{T}}$, and likewise $S_{ij}$ instead of $\overline{T}_{ij}$, for the corresponding matrix and holor representations after transformation. The tensor is invariably denoted by the symbol $T \in \mathbf{T}^0_2(V)$. In particular,

$$T = T_{ij} \hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}^j = S_{ij} \hat{\mathbf{f}}^i \otimes \hat{\mathbf{f}}^j \,.$$

(5) **b.** Show that $T_{k\ell} = B^i_k B^j_\ell S_{ij}$.

(5) **c.** Show that $\det \mathbf{T}$ is not a scalar by showing that $\det \mathbf{S} = (\det \mathbf{A})^2 \det \mathbf{T}$.

(5) **d.** Show that, consequently, the collection of numbers $\tilde{S}^{ij}$ does not transform as the holor of a rank-2 contratensor, and derive the correct “relative tensor transformation law” for it.

Below we assume that $V$ is equipped with a real inner product, with $(\mathbf{x}|\mathbf{y}) = G(\mathbf{x},\mathbf{y}) = g_{ij}x^i y^j$, in which $g_{ij} = (\mathbf{e}_i|\mathbf{e}_j)$ are the corresponding components of the Gram matrix $\mathbf{G}$ relative to basis $\{\mathbf{e}_i\}$, with $g = \det \mathbf{G}$.

(5) **e.** We stipulate that $V^{ij} = g^w \tilde{T}^{ij}$, for some $w \in \mathbb{R}$, is the holor of a rank-2 contratensor that does adhere to the usual tensor transformation law. Prove that such a $w \in \mathbb{R}$ exists and determine its value.

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**(25)** **4.** LOG-POLAR RETINA.

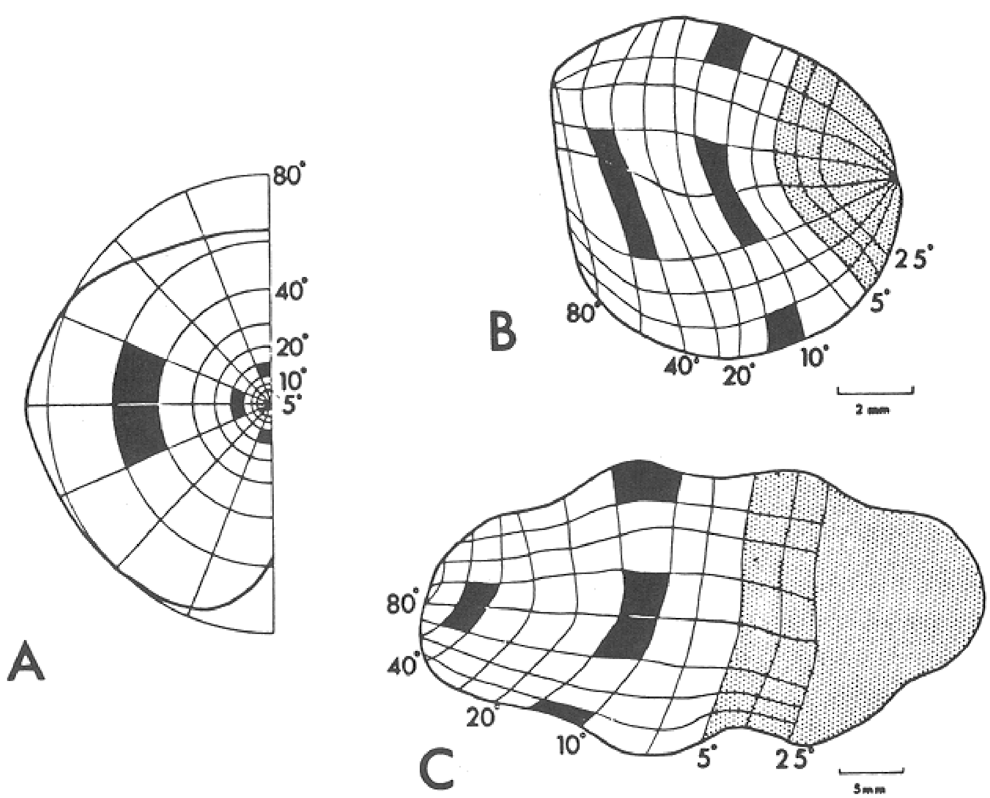
We consider a simplified model of the human retina in the form of a 2-dimensional manifold with a basic volume form $\boldsymbol{\mu} \in \bigwedge_2(\mathbb{R}^2)$ given in polar coordinates by

$$\boldsymbol{\mu} = \frac{1}{r}\, dr \wedge d\phi\,.$$

Recall the definition of polar coordinates $(r, \phi)$ in terms of Cartesian coordinates $(x, y)$:

$$\begin{cases} x &= r\cos\phi \\ y &= r\sin\phi \end{cases}$$

(We ignore subtleties that normally need to be considered in the context of reparametrizations, such as injectivity, differentiability and invertibility.)




MAPPING (CONNOLLY AND VAN ESSEN 1984) OF THE VISUAL FIELD (A) ON THE LGN (B) AND THE STRIATE CORTEX (C) IN MONKEY. THE REPRESENTATION OF THE CENTRAL 5 DEGREES (SHADED AREAS) IN THE VISUAL FIELD OCCUPIES ABOUT 40% OF THE CORTEX. THE VOLUME FORM $\boldsymbol{\mu}$ IN THIS PROBLEM ANATOMICALLY REPRESENTS A DIFFERENTIAL SURFACE ELEMENT ON THE HUMAN STRIATE CORTEX RETINOTOPICALLY CONNECTED TO A CONSTANT SURFACE ELEMENT $dx \wedge dy$ ON THE RETINA AT POSITION $(x, y)$ RELATIVE TO THE FOVEAL CENTER. PERCEPTUALLY IT EXPLAINS THE LINEAR DECREASE OF RESOLVING POWER AS A FUNCTION OF ECCENTRICITY $r$.

(5) **a1.** Express the 2-form $\boldsymbol{\mu}$ in $(x, y)$ coordinates.

(5) **a2.** Find "log-polar coordinates" coordinates $(\rho, \psi)$ such that $\boldsymbol{\mu} = d\rho \wedge d\psi$.

**Definition.** The Christoffel symbols $\Gamma^k_{ij}$ associated with the affine connection $\nabla : \mathrm{T}\mathbb{R}^2 \times \mathrm{T}\mathbb{R}^2 \to \mathrm{T}\mathbb{R}^2$ are defined by $\nabla_{\partial_j} \partial_i = \Gamma^k_{ij} \partial_k$ $(i = 1, 2)$.

(5) **b.** Show that the components of the Christoffel symbols in a coordinate system $\overline{\mathbf{x}}$ are given relative to those in a coordinate system $\mathbf{x}$ by

$$\overline{\Gamma}^k_{ij} = S^k_\ell \left( T^m_j T^n_i \Gamma^\ell_{nm} + \overline{\partial}_j T^\ell_i \right) ,$$

in which $\mathbf{T}$ and $\mathbf{S}$ are the Jacobian matrices with components

$$T^i_j = \frac{\partial x^i}{\partial \overline{x}^j} \quad \text{resp.} \quad S^i_j = \frac{\partial \overline{x}^i}{\partial x^j} .$$

(5) **c.** Under the assumption that $\Gamma^k_{ij} = 0$ in Cartesian coordinates $(x, y)$, evaluate the induced symbols $\overline{\Gamma}^k_{ij}$ in log-polar coordinates $(\rho, \psi)$.

(5) **d.** As opposed to the previous problem we now assume that $\overline{\Gamma}^k_{ij} = 0$ in log-polar coordinates $(\rho, \psi)$. Evaluate the induced symbols $\Gamma^k_{ij}$ in Cartesian coordinates $(x, y)$.

**THE END**