EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2F800. Date: Wednesday June 26, 2013. Time: 14h00–17h00. Place:

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- You may consult the course notes “Tensorrekening en Differentiaalmeetkunde (2F800)” by Jan de Graaf, and the draft notes “Tensor Calculus and Differential Geometry (2F800)” by Luc Florack without warranty.

(**35**) **1.** MAXWELL EQUATIONS IN MINKOWSKI SPACETIME.

Definition. We consider “empty” 4-dimensional Minkowski spacetime, furnished with a Lorentzian metric $(\mathbf{x}|\mathbf{y}) = G(\mathbf{x}, \mathbf{y}) = g_{ij}x^i y^j$ (on each tangent space $V = \mathrm{TM}_x \sim \mathbb{R}^4$), in which $g_{ij} = (\partial_i|\partial_j)$ are the components of the corresponding Gram matrix $\mathbf{G}$. The construct is similar to that of a Riemannian manifold, except that the metric fails to satisfy the positivity axiom of a Riemannian inner product. In Cartesian coordinates we have $g_{11} = g_{22} = g_{33} = 1$, $g_{44} = -1$, $g_{ij} = 0$ otherwise ($i, j = 1, 2, 3, 4$).

We simplify notation: $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\partial_3 = \partial_z$, $\partial_4 = \partial_t$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$, $dx^4 = dt$.

Definition. The converters $\sharp : V \to V^*$ and $\flat : V^* \to V$ are defined as follows. If $\mathbf{v} \in V$, $\hat{\mathbf{v}} \in V^*$, then

$$\sharp\mathbf{v} \stackrel{\text{def}}{=} G(\mathbf{v}, \,\cdot\,) \in V^* \quad \text{respectively} \quad \flat\hat{\mathbf{v}} \stackrel{\text{def}}{=} G^{-1}(\hat{\mathbf{v}}, \,\cdot\,) \in V\,.$$

Definition. Let $\hat{\mathbf{a}}^1, \ldots, \hat{\mathbf{a}}^k \in V^*$. The Hodge star operator $* : \bigwedge_k(V) \to \bigwedge_{n-k}(V)$ is defined as follows:

$$\begin{cases} *1 & = \boldsymbol{\epsilon} & \text{for } k = 0, \\ *\left(\hat{\mathbf{a}}^1 \wedge \ldots \wedge \hat{\mathbf{a}}^k\right) & = \boldsymbol{\epsilon} \lrcorner \flat\hat{\mathbf{a}}^1 \lrcorner \ldots \lrcorner \flat\hat{\mathbf{a}}^k & \text{for } k = 1, \ldots, n. \end{cases}$$

In this definition $\boldsymbol{\epsilon} = \sqrt{|g|}\, dx^1 \wedge \ldots \wedge dx^n = \epsilon_{|i_1 \ldots i_n|} dx^{i_1} \wedge \ldots \wedge dx^{i_n} \in \bigwedge_n(V)$ is the Levi-Civita tensor, with $\epsilon_{1\ldots n} = \sqrt{|g|} = \sqrt{|\det \mathbf{G}|}$.

Definition. Let $\mathbf{a}_1, \ldots, \mathbf{a}_k \in V$. The $(n-k)$-form $\boldsymbol{\epsilon} \lrcorner \mathbf{a}_1 \lrcorner \ldots \lrcorner \mathbf{a}_k \in \bigwedge_{n-k}(V)$ is defined as follows:

$$(\boldsymbol{\epsilon} \lrcorner \mathbf{a}_1 \lrcorner \ldots \lrcorner \mathbf{a}_k)(\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n) = \boldsymbol{\epsilon}(\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n) \quad \text{for all } \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n \in V.$$

(10) **a.** Compute the following forms in terms of wedge products of dx, dy, dz, and dt:

- $*\,1$;
- $*\,dx$, $*\,dy$, $*\,dz$, $*\,dt$.

Lemma. Without proof the following identities are provided for your convenience.

- $*(dx \wedge dy) = dz \wedge dt$;
- $*(dx \wedge dz) = -dy \wedge dt$;
- $*(dx \wedge dt) = -dy \wedge dz$;
- $*(dy \wedge dz) = dx \wedge dt$;
- $*(dy \wedge dt) = dx \wedge dz$;
- $*(dz \wedge dt) = -dx \wedge dy$.

Definition. We define the *Faraday* 2-form $F(x,y,z,t) = E(x,y,z,t) + B(x,y,z,t) \in \bigwedge_2(V)$, with

$$\begin{aligned} E(x,y,z,t) &= -E_1(x,y,z,t)\, dx \wedge dt - E_2(x,y,z,t)\, dy \wedge dt - E_3(x,y,z,t)\, dz \wedge dt\,, \\ B(x,y,z,t) &= -B_1(x,y,z,t)\, dy \wedge dz + B_2(x,y,z,t)\, dx \wedge dz - B_3(x,y,z,t)\, dx \wedge dy\,. \end{aligned}$$

The independent components of $E(x,y,z,t)$ and $B(x,y,z,t)$ are collected into column 3-vectors:

$$\boldsymbol{E}(x,y,z,t) \stackrel{\text{def}}{=} \begin{pmatrix} E_1(x,y,z,t) \\ E_2(x,y,z,t) \\ E_3(x,y,z,t) \end{pmatrix} \quad \text{and} \quad \boldsymbol{B}(x,y,z,t) \stackrel{\text{def}}{=} \begin{pmatrix} B_1(x,y,z,t) \\ B_2(x,y,z,t) \\ B_3(x,y,z,t) \end{pmatrix}.$$

(10) **b.** Suppose $F = F_{\mu\nu}\, dx^\mu \otimes dx^\nu = F_{|\mu\nu|}\, dx^\mu \wedge dx^\nu$. Provide the matrix $\boldsymbol{F}$ with entries $F_{\mu\nu}$.

Definition. Let $\omega^k(x,y,z,t) = \omega_{|i_1 \ldots i_k|}(x,y,z,t)\, dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \bigwedge_k(V)$ be a k-form on V. The *exterior derivative* of $\omega^k(x)$ is the $(k+1)$-form given by

$$d\omega^k(x,y,z,t) = d\omega_{|i_1 \ldots i_k|}(x,y,z,t) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \bigwedge_{k+1}(V)\,,$$

with $df(x,y,z,t) \stackrel{\text{def}}{=} \partial_i f(x,y,z,t)\, dx^i$ for a sufficiently smooth scalar field $f : \mathbb{R}^4 \to \mathbb{R}$.

Definition. Recall the first order derivative operators div and rot defined for 3-vector valued functions on (x,y,z)-space (t is treated as a parameter and is irrelevant here):

$$\operatorname{div} X \stackrel{\text{def}}{=} \partial_x X_1 + \partial_y X_2 + \partial_z X_3 \quad \text{and} \quad \operatorname{rot} X \stackrel{\text{def}}{=} \begin{pmatrix} \partial_y X_3 - \partial_z X_2 \\ \partial_z X_1 - \partial_x X_3 \\ \partial_x X_2 - \partial_y X_1 \end{pmatrix}.$$

($7\frac{1}{2}$) **c.** Show that $dF = 0$ is equivalent to the following system of pde's for $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\operatorname{rot} \boldsymbol{E} \quad \text{and} \quad \operatorname{div} \boldsymbol{B} = 0\,.$$

Definition. The *Maxwell* 2-form $*F$ is the Hodge dual of the Faraday 2-form F.

($7\frac{1}{2}$) **d.** Show that $d * F = 0$ is equivalent to the following system of pde's for $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\frac{\partial \boldsymbol{E}}{\partial t} = \operatorname{rot} \boldsymbol{B} \quad \text{and} \quad \operatorname{div} \boldsymbol{E} = 0\,.$$

(*Hint:* A symmetry consideration after computation of $*F$ may considerably simplify your proof.)

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(**35**) **2.** DIVERGENCE.

We consider an n-dimensional Riemannian manifold M. The holor of the metric tensor is given by $g_{ij} = (\partial_i|\partial_j)$ relative to the local coordinate basis $\{\partial_i\}$ of TM_x. Recall that the Levi-Civita connection induces a covariant derivative compatible with the metric, implying the (dual) metric to be "covariantly constant": $D_i g_{jk} = 0$ and $D_i g^{jk} = 0$.

Let $v = v^i \partial_i$ be a smooth vector field on M. We wish to define its *divergence* in a geometrically meanigful, i.e. coordinate independent way.

(5) **a.** Show that the "standard" definition $\operatorname{div} v \stackrel{\text{def}}{=} \partial_i v^i$ fails to be coordinate independent.
(*Tip:* The solution to problem b implies the solution to this problem.)

We stipulate a *covariant divergence* of the form $D_i v^i \stackrel{\text{def}}{=} (\partial_i + A_i) v^i$ for some "gauge field" $A = A_i dx^i$. In the next problem this definition is not necessarily compatible with the metric, i.e. the field A is not necessarily related to the Levi-Civita connection.

(5) **b.** Show that the gauge field does *not* transform as a covector field, and derive its explicit transformation behaviour if we assume $\operatorname{div} v \stackrel{\text{def}}{=} D_i v^i$ to be coordinate independent.

Henceforth we will assume the covariant derivative to be compatible with the metric. Recall the Christoffel symbols of the second kind for the Levi-Civita connection:

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} \left(\partial_j g_{\ell k} + \partial_k g_{j\ell} - \partial_\ell g_{jk} \right)\,.$$

(5) **c.** Prove that $\Gamma^i_{ik} = \Gamma^i_{ki} = \partial_k \ln \sqrt{g}$.
(*Hint:* Recall the definition of the cofactor matrix and its relation to the matrix inverse.)

Lemma. If J is the determinant of the Jacobian matrix with entries $\dfrac{\partial x^k}{\partial \overline{x}^i}$, then $\overline{\partial}_k J = \dfrac{\partial \overline{x}^j}{\partial x^\ell} \dfrac{\partial^2 x^\ell}{\partial \overline{x}^k \partial \overline{x}^j} J$.

(5) **d.** Show that $A_k \stackrel{\text{def}}{=} \Gamma^i_{ik}$ is consistent in the sense that this gauge field transforms exactly as in b.
(*Hint:* You might need the lemma.)

(5) **e.** Prove the identity $D_i v^i = \dfrac{1}{\sqrt{g}} \partial_i \left(\sqrt{g}\, v^i \right)$.

(*Hint:* Use c and d.)

Let f be a smooth scalar field on M. Its *vector gradient* is defined relative to the local coordinate basis $\{\partial_i\}$ of TM_x as grad $f \stackrel{\text{def}}{=} g^{ij}\partial_j f\,\partial_i$.

(5) **f.** Show that this is a coordinate independent definition.

For this reason we define the holor of grad f as $D^i f \stackrel{\text{def}}{=} g^{ij}\partial_j f$.

Let f be a smooth scalar field on M. We define its *Laplacian* as $\Delta f \stackrel{\text{def}}{=} D_i D^i f$.

(5) **g.** Prove the identity $D_i D^i f = \dfrac{1}{\sqrt{g}}\,\partial_i\left(\sqrt{g}\,g^{ij}\partial_j f\right)$.

(*Hint:* Use e.)

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(**30**) **3.** GRAVITATIONAL WAVES*.

A wave propagating in n-dimensional spacetime ($n \geq 2$) with unit velocity $c = 1$ in z-direction is described by a wave function $h(z,t)$ satisfying the wave equation

$$\frac{\partial^2 h}{\partial t^2} - \frac{\partial^2 h}{\partial z^2} = 0\,.$$

The additional $n-2$ spatial coordinates x and y are treated as parameters and are, for the moment, suppressed in the notation. We introduce the light-cone coordinates

$$u \stackrel{\text{def}}{=} \frac{t-z}{\sqrt{2}} \quad \text{and} \quad v \stackrel{\text{def}}{=} \frac{t+z}{\sqrt{2}}\,.$$

(5) **a.** Define $H(u,v) \stackrel{\text{def}}{=} h(z,t)$. Transform the (z,t)-p.d.e. for h into a (u,v)-p.d.e. for H and show that it admits traveling wave solutions of the form $H(u,v) = F(u) + G(v)$.

Under certain conditions such traveling waves turn out to be compatible with Einstein's field equations for the gravitational field, which in turn defines the (Levi-Civita connection of the) Lorentzian spacetime metric. One exact solution in $n=4$ (with additional spatial coordinates indicated by x, y) is given by the so-called pp-wave metric

$$G = -du\otimes dv - dv\times du - 2H(u,x,y)\,du\otimes du + dx\otimes dx + dy\otimes dy\,. \qquad (\ddagger)$$

Note that H does not depend on v. The only non-vanishing Christoffel symbols are $\Gamma^x_{uu} = \Gamma^v_{xu} = \Gamma^v_{ux}$, $\Gamma^y_{uu} = \Gamma^v_{yu} = \Gamma^v_{uy}$, and Γ^v_{uu}.

(10) **b.** Derive the formulas for these connection symbols in terms of $H(u,x,y)$.

*CF. A. FUSTER. "KUNDT SPACETIMES IN GENERAL RELATIVITY AND SUPERGRAVITY", © 2007, VU UNIVERSITY AMSTERDAM.

The holor of the Riemann tensor is given by

$$R^{\lambda}{}_{\mu\rho\nu} = \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\mu\rho}\,.$$

Alternatively one defines the fully covariant Riemann tensor, with holor

$$R_{\lambda\mu\rho\nu} = g_{\lambda\sigma} R^{\sigma}{}_{\mu\rho\nu}\,.$$

The holor of the Ricci tensor follows by a contraction:

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}\,.$$

The Ricci scalar is given by

$$R = g^{\mu\nu} R_{\mu\nu}\,.$$

Einstein's field equations for the metric $g_{\mu\nu}$ in vacuum (without cosmological constant) are given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0\,.$$

(5) **c.** Show that for any vacuum solution $g_{\mu\nu}$ of these equations we necessarily have $R = 0$, reducing the field equations to $R_{\mu\nu} = 0$.

Without proof we provide the only independent non-vanishing components of $R_{\lambda\mu\rho\nu}$ given the metric stipulated above, viz.

$$R_{uxux} = -\partial_{xx} H\ ,\ R_{uyuy} = -\partial_{yy} H\ ,\ R_{uxuy} = -\partial_{xy} H\,.$$

(10) **d.** Show that the stipulated metric (‡) is an exact vacuum solution of Einstein's field equations if H is a harmonic function in the (x, y)-plane, i.e. if $\Delta H \stackrel{\text{def}}{=} \partial_{xx} H + \partial_{yy} H = 0$.
(*Hint:* Show that R_{uu} is the only non-zero component of the Ricci tensor, and compute its value.)

THE END