INTERIM TEST TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0-2WAH2. Date: Tuesday February 24, 2015. Time: 15h45–17h30. Place: Flux 1.07.

Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 1 \\ 6 & 3 & 8 \\ 8 & 7 & 10 \end{pmatrix}$. Use these matrices in problems 1–5.

1. Compute (i) $\det A$, (ii) $\det B$, (iii) perm A, and (iv) perm B.

$\det A = 1$, $\det B = 0$, perm $A = 13$, perm $B = 400$.

2. Compute the following expression involving the standard 3-dimensional inner and outer product of vectors (cf. the columns—$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, say—of B), and provide a geometrical interpretation of your result for $\det B$ in problem 1 (ii):

$$(\mathbf{b}_1 \times \mathbf{b}_2) \cdot \mathbf{b}_3 = \left[\begin{pmatrix} 1 \\ 6 \\ 8 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 8 \\ 10 \end{pmatrix}$$

This expression yields 0, indicating that the third column vector of B lies within the 2-dimensional plane spanned by the first two column vectors. This explains why $\det B = 0$. In fact, $\det B = (\mathbf{b}_1 \times \mathbf{b}_2) \cdot \mathbf{b}_3$.

3. Compute the following cofactor and adjugate matrices: (i) $\tilde{A}$, (ii) $\tilde{B}$, (iii) $\tilde{A}^{\mathrm{T}}$, and (iv) $\tilde{B}^{\mathrm{T}}$.

$$\tilde{A} = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} -26 & 4 & 18 \\ -13 & 2 & 9 \\ 13 & -2 & -9 \end{pmatrix}, \quad \tilde{A}^T = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}, \quad \tilde{B}^T = \begin{pmatrix} -26 & -13 & 13 \\ 4 & 2 & -2 \\ 18 & 9 & -9 \end{pmatrix}.$$

4. Compute the following inverse matrices, if these exist: (i) A^{-1} and (ii) B^{-1}.

Since $\det A = 1$ we have $A^{-1} = \dfrac{1}{\det A}\tilde{A}^T = \tilde{A}^T$, cf. previous problem. Since $\det B = 0$ the matrix B^{-1} does not exist.

5. Compute (i) $A\tilde{A}^{\mathrm{T}}$ and (ii) $B\tilde{B}^{\mathrm{T}}$.

We have $A\tilde{A}^T = \det A I = I$, i.e. the 2×2 identity matrix, respectively $B\tilde{B}^T = \det B I = 0$, i.e. the 3×3 null matrix.

6. Consider the collection $\{A_{i_1 \dots i_n} \in \mathbb{R} \,|\, i_1, \dots, i_n = 1, \dots, n\}$.

a. How many independent components does this collection have if no constraints are imposed?

There are n indices, each of which can assume n distinct values, whence there are a priori n^n independent components.

b. Suppose $A_{i_1 \dots i_k \dots i_\ell \dots i_n} = -A_{i_1 \dots i_\ell \dots i_k \dots i_n}$ for *any* $1 \leq k < \ell \leq n$ (complete antisymmetry). Show that $A_{i_1 \dots i_n} \propto [i_1 \dots i_n]$, in which the symbol $\propto$ indicates proportionality.

From complete antisymmetry under each index exchange it follows that all symbols with two or more equal index labels (say $i_k = i_\ell$) must necessarily vanish (as 0 is the only number equal to its opposite). This implies that for nontrivial symbols $A_{i_1\ldots i_n}$ the index set $(i_1, \ldots, i_n)$ must be a permutation of $(1, \ldots, n)$. Setting $A_{1\ldots n} = C$ for some constant C, all $A_{i_1\ldots i_n}$ can be obtained as $\pm C$ through repetitive index exchanges exploiting the antisymmetry property, viz. $A_{i_1\ldots i_n} = C[i_1 \ldots i_n]$. Thus the underlying "hypermatrix" A, comprising all n^n entries $A_{i_1\ldots i_n}$, has in fact only a single degree of freedom.

7. Compute the multi-dimensional Gaussian integral $\gamma(A,s) = \displaystyle\int_{\mathbb{R}^n} \exp(-x^i A_{ij} x^j + s_k x^k)\, dx$ for a symmetric positive definite $n \times n$ matrix A and "source" covector $s \in \mathbb{R}^n$. Here $dx = dx^1 \ldots dx^n$.

You may make use of the following facts:

- $\displaystyle\int_{-\infty}^{\infty} \exp(-az^2)\, dz = \sqrt{\frac{\pi}{a}}$ for $a > 0$.
- There exists a rotation matrix R such that $R^{\mathrm{T}} A R = \Delta$, with $\Delta = \mathrm{diag}\,(\lambda_1 > 0, \ldots, \lambda_n > 0)$.

I. First consider the case $s = 0 \in \mathbb{R}^n$, i.e.

$$\gamma(A,0) = \int_{\mathbb{R}^n} \exp(-x^i A_{ij} x^j)\, dx\,.$$

A symmetric matrix can be diagonalized via a rotation. Thus consider the following substitution of variables: $x = Ry$, whence $x^T = y^T R^T$ (note that $R^T = R^{-1}$) and $dx = \det R\, dy = dy$. Define the diagonal matrix $\Delta = R^T A R = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, with $\lambda_p > 0$ for all $p = 1, \ldots, n$ by virtue of positive definiteness. Then, with parameters λ_p considered implicitly as a function of matrix A as indicated,

$$\gamma(A,0) = \prod_{p=1}^{n} \int_{\mathbb{R}} \exp(-\lambda_p (y^p)^2)\, dy^p\,.$$

If desired, one may absorb the eigenvalues λ_p into the integration variables by another substitution of variables in order to further simplify the result, viz. consider the rescaling $z^p = \sqrt{\lambda_p} y^p$. Then, again assuming λ_p to be given as functions of A,

$$\gamma(A,0) = \prod_{p=1}^{n} \frac{1}{\sqrt{\lambda_p}} \int_{\mathbb{R}} \exp(-(z^p)^2)\, dz^p\,.$$

This reduces the problem to a standard one-dimensional Gaussian integral: $\displaystyle\int_{\mathbb{R}} \exp(-z^2)\, dz = \sqrt{\pi}$. Note that $\prod_{p=1}^{n} \lambda_p = \det A$, so that

$$\gamma(A,0) = \frac{\sqrt{\pi}^n}{\sqrt{\det A}}\,.$$

II. Next consider the case for a nontrivial source covector $s \in \mathbb{R}^n$, i.e.

$$\gamma(A,s) = \int_{\mathbb{R}^n} \exp(-x^i A_{ij} x^j + s_k x^k)\, dx\,.$$

Rewrite the exponent in the integrand as a quadratic form in $x - \xi$ for some vector $\xi \in \mathbb{R}^n$ and an x-independent (ξ-dependent) remainder, and solve for ξ in terms of s and A. In this way one obtains $\xi^i = \frac{1}{2} s_k B^{ki}$, in which $B = A^{-1}$, i.e. $A_{ij} B^{jk} = \delta_i^k$, so that

$$\exp(-x^i A_{ij} x^j + s_k x^k) = \exp\left(-(x^i - \frac{1}{2} s_k B^{ki})\, A_{ij}\, (x^j - \frac{1}{2} s_\ell B^{\ell j})\right) \exp\left(\frac{1}{4} s_p B^{pq} s_q\right)\,,$$

With the help of a substitution of variables, $y^i = x^i - \frac{1}{2} s_k B^{ki}$, for which $dy = dx$, we then obtain

$$\gamma(A,s) = \exp\left(\frac{1}{4} s_p B^{pq} s_q\right) \int_{\mathbb{R}^n} \exp\left(-y^i\, A_{ij}\, y^j)\right)\, dy = \frac{\sqrt{\pi}^n}{\sqrt{\det A}} \exp\left(\frac{1}{4} s_p B^{pq} s_q\right) = \frac{\sqrt{\pi}^n}{\sqrt{\det A}} \exp\left(\frac{1}{4} s^T A^{-1} s\right)\,.$$

THE END