INTERIM TEST 2WAH0: TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH2. Date: Tuesday February 24, 2015. Time: 15h45–17h30. Place: Flux 1.07.

☞ The following matrices are used in problems 1–5: $A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 1 \\ 6 & 3 & 8 \\ 8 & 7 & 10 \end{pmatrix}$.

(2) **1.** Compute (i) $\det A$, (ii) $\det B$, (iii) perm A, and (iv) perm B.

(1) **2.** Compute the following expression involving the standard 3-dimensional inner and outer product of vectors (cf. the columns—$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, say—of B), and provide a geometrical interpretation of your result for $\det B$ in problem 1 (ii):

$$(\mathbf{b}_1 \times \mathbf{b}_2) \cdot \mathbf{b}_3 = \left[\begin{pmatrix} 1 \\ 6 \\ 8 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 8 \\ 10 \end{pmatrix} .$$

(2) **3.** Compute the following cofactor and adjugate matrices: (i) $\tilde{A}$, (ii) $\tilde{B}$, (iii) $\tilde{A}^{\mathrm{T}}$, and (iv) $\tilde{B}^{\mathrm{T}}$.

(1) **4.** Compute the following inverse matrices, if these exist: (i) A^{-1} and (ii) B^{-1}.

(1) **5.** Compute (i) $A\tilde{A}^{\mathrm{T}}$ and (ii) $B\tilde{B}^{\mathrm{T}}$.

(2) **6.** Consider the collection $\{A_{i_1 \ldots i_n} \in \mathbb{R} \,|\, i_1, \ldots, i_n = 1, \ldots, n\}$.

a. How many independent components does this collection have if no constraints are imposed?

b. Suppose $A_{i_1 \ldots i_k \ldots i_\ell \ldots i_n} = -A_{i_1 \ldots i_\ell \ldots i_k \ldots i_n}$ for *any* $1 \leq k < \ell \leq n$ (complete antisymmetry). Show that $A_{i_1 \ldots i_n} \propto [i_1 \ldots i_n]$, in which the symbol $\propto$ indicates proportionality.

(4) **7.** Compute the multi-dimensional Gaussian integral $\gamma(A, s) = \displaystyle\int_{\mathbb{R}^n} \exp(-x^i A_{ij} x^j + s_k x^k)\, dx$ for a symmetric positive definite $n \times n$ matrix A and "source" covector $s \in \mathbb{R}^n$. Here $dx = dx^1 \ldots dx^n$.

☞ You may use the following lemmas:

- $\displaystyle\int_{-\infty}^{\infty} \exp(-az^2)\, dz = \sqrt{\frac{\pi}{a}}$ for $a > 0$.
- There exists a rotation matrix R such that $R^{\mathrm{T}} A R = \Delta$, with $\Delta = \mathrm{diag}\,(\lambda_1 > 0, \ldots, \lambda_n > 0)$.

(2) **8.** Suppose $\{\mathbf{e}_i\}$ and $\{\hat{\mathbf{e}}^i\}$ are dual bases of V and V^*, and $\mathbf{e}_i = A_i^j \mathbf{f}_j$, $\hat{\mathbf{e}}^i = B_j^i \hat{\mathbf{f}}^j$ for some a priori unrelated transformation matrices A and B. Show that if $\{\mathbf{f}_j, \hat{\mathbf{f}}^j\}$ constitute dual bases then $B_k^j A_i^k = \delta_i^j$.

THE END