INTERIM TEST 2WAH0: TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH2. Date: Thursday February 25, 2016. Time: 10:45–11:30. Place: Paviljoen a 12 b.

(**30**) **1.** Let $\mathbb{M}^n$ denote the set of all $n \times n$-matrices with real matrix entries. By cof A and adj A we denote the *cofactor matrix*, resp. the *adjugate matrix* of A. These are defined as follows:

Definition. If $A \in \mathbb{M}^n$ has matrix entries A_{ij}, then $(\text{cof } A)^{ij} = \dfrac{\partial \det A}{\partial A_{ij}}$, resp. $(\text{adj } A)^{ij} = \dfrac{\partial \det A}{\partial A_{ji}}$.

In all that follows you may use the following lemma.

Lemma. If $\det A \neq 0$, then adj $A = \det A \, A^{-1}$.

We consider a parametrized curve in $\mathbb{M}^n$, i.e. a smooth matrix-valued function of one variable:

$$A : \mathbb{R} \to \mathbb{M}^n : t \mapsto A(t) \, .$$

Its component functions are indicated as $A_{ij} : \mathbb{R} \to \mathbb{R} : t \mapsto A_{ij}(t)$.

(5) **a.** Use the definition to prove the following theorem, in which tr $: \mathbb{M}^n \to \mathbb{R} : X \mapsto \text{tr}\,(X)$ stands for the trace operator, i.e. summation of diagonal elements: if X has components X^i_j, then $\text{tr}\,(X) = X^i_i$ (summation convention):

$$\frac{d \det A(t)}{dt} = \text{tr} \left(\text{adj } A(t) \, \frac{dA(t)}{dt} \right) \, .$$

(5) **b.** Show that if $\det A(t) \neq 0$ for all $t \in \mathbb{R}$, then

$$\frac{d \log \left(\det A(t)\right)}{dt} = \text{tr} \left(A^{-1}(t) \, \frac{dA(t)}{dt} \right) \, .$$

The *matrix exponential* Exp X of a matrix $X \in \mathbb{M}^n$ is defined in terms of a series expansion:

$$\text{Exp} \, : \mathbb{M}^n \to \mathbb{R} : X \mapsto \text{Exp}\,(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k \, ,$$

analogous to the Taylor series of its numeric counterpart,

$$\exp : \mathbb{R} \to \mathbb{R} : x \mapsto \exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \, .$$

Integer matrix powers are defined as repetitive matrix autoproducts, with, by definition, $X^0 = I$.

We now restrict ourselves to linear functions, $A(t) = t\,X$ for some constant matrix $X \in \mathbb{M}^n$.

(5) **c.** Show hat

$$\frac{d\,\mathrm{Exp}\,(t\,X)}{dt} = \mathrm{Exp}\,(t\,X)\,X\,.$$

In subsequent questions you may use the following lemma (a special case of the CBH-formula).

Lemma. If $X, Y \in \mathbb{M}^n$ are commuting matrices (i.e. $XY = YX$), then $\mathrm{Exp}\,(X+Y) = \mathrm{Exp}\,(X)\,\mathrm{Exp}\,(Y)$.

(5) **d.** Show, using the lemma, that $\mathrm{Exp}\,(X) \in \mathbb{M}^n$ is nonsingular regardless of $X \in \mathbb{M}^n$, and

$$(\mathrm{Exp}\,(X))^{-1} = \mathrm{Exp}\,(-X)\,.$$

(5) **e.** Let $u : \mathbb{R} \to \mathbb{R} : t \mapsto u(t) = \det \mathrm{Exp}\,(tX)$. Show that this function satisfies the differential equation

$$\begin{cases} \dfrac{du}{dt} &= \mathrm{tr}\,(X)\,u\,, \\ u(0) &= 1\,. \end{cases}$$

(5) **f.** Show that

$$\det \mathrm{Exp}\,(X) = \exp\left(\mathrm{tr}\,(X)\right)\,.$$

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☞ It is useful to memorize “$\det \mathrm{Exp} = \exp \mathrm{tr}$”.

THE END