**Problem Companion & Solutions** 

# **Tensor Calculus and Differential Geometry**

2WAH0



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Cover illustration: papyrus fragment from Euclid's Elements of Geometry, Book II [8].

# Contents

Preface		1
1	Prerequisites from Linear Algebra	3
2	Tensor Calculus	7
3	Differential Geometry	15

## Preface

This problem companion belongs to the course notes "Tensor Calculus and Differential Geometry" (course code 2WAH0) by Luc Florack. Problems are concisely formulated. Einstein summation convention applies to all problems, unless stated otherwise. Please refer to the course notes for further details.

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## 1. Prerequisites from Linear Algebra

**1.** Let V be a real vector space, and  $u \in V$ . Show that  $(-1) \cdot u = -u$ , and  $0 \cdot u = o$ .

Each step in the following derivations either exploits a basic vector space axiom or reflects a trivality:

- $u + 0 \cdot u = 1 \cdot u + 0 \cdot u = (1 + 0) \cdot u = 1 \cdot u = u$  for all  $u \in V$ , whence  $0 \cdot u = o \in V$ , i.e. the null vector;
- $u + (-1) \cdot u = 1 \cdot u + (-1) \cdot u = (1 + (-1)) \cdot u = 0$  for all  $u \in V$  according to the previous result, whence  $(-1) \cdot u = -u$ .

#### **2.** Show that $\mathscr{L}(V, W)$ is a vector space.

Let  $\mathscr{A}, \mathscr{B} \in \mathscr{L}(V, W)$ , then  $\lambda \mathscr{A} + \mu \mathscr{B} \in \mathscr{L}(V, W)$  is the linear operator defined by  $(\lambda \mathscr{A} + \mu \mathscr{B})(v) = \lambda \mathscr{A}(v) + \mu \mathscr{B}(v)$  for all  $v \in V$ . The proof of the vector space structure of  $\mathscr{L}(V, W)$  exploits the vector space structure of W, as follows. Let  $v \in V$  be any given vector,  $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathscr{L}(V, W)$ , and consider, say, the associativity axiom:  $((\mathscr{A} + \mathscr{B}) + \mathscr{C})(v) = (\mathscr{A} + \mathscr{B})(v) + \mathscr{C}(v) = \mathscr{A}(v) + \mathscr{B}(v) + \mathscr{C}(v) = \mathscr{A}(v) + (\mathscr{B} + \mathscr{C})(v) = (\mathscr{A} + (\mathscr{B} + \mathscr{C}))(v)$ , whence  $(\mathscr{A} + \mathscr{B}) + \mathscr{C} = \mathscr{A} + (\mathscr{B} + \mathscr{C})$ . Note that triple sums on W do not require disambiguating parentheses as a result of its vector space structure. The remaining seven axioms are proven in a similar fashion by evaluation on an arbitrary dummy  $v \in V$ .

In analogy with the determinant of a matrix A, the so-called *permanent* is defined as

$$\operatorname{perm} A = \sum_{j_1, \dots, j_n = 1}^n |[j_1, \dots, j_n]| A_{1j_1} \dots A_{nj_n} = \frac{1}{n!} \sum_{\substack{i_1, \dots, i_n = 1 \\ j_1, \dots, j_n = 1}}^n |[i_1, \dots, i_n] [j_1, \dots, j_n]| A_{i_1 j_1} \dots A_{i_n j_n} A_{i$$

Note that the *nontrivial* factors among the weights  $|[i_1, \ldots, i_n]|$  and  $|[j_1, \ldots, j_n]|$  are invariably +1.

**3.** Can we omit the factors  $|[i_1, \ldots, i_n]|$  and  $|[j_1, \ldots, j_n]|$  in this definition of perm A?

No. They are needed to effectively constrain the  $(i_1, \ldots, i_n)$  and  $(j_1, \ldots, j_n)$  sums to all n! permutations of  $(1, \ldots, n)$ . Without these factors the sums would effectively have  $n^n \ge n!$  terms.

4. How many multiplications and additions/subtractions do you need in the numerical computation of det A and perm A for a generic  $n \times n$  matrix A?

For both det A and perm A we generically have n! nontrivial terms, each consisting of an n-fold product of matrix entries. This implies that we need n! - 1 additions/subtractions of terms, each of which requires n - 1 multiplications.

5. Given a generic  $n \times n$  matrix A. Argue why its cofactor and adjugate matrices  $\tilde{A}$ , respectively  $\tilde{A}^{T}$ , *always* exist, unlike its inverse  $A^{-1}$ . What is the condition for  $A^{-1}$  to be well-defined?

By construction the entries of the cofactor and adjugate matrices are homogeneous (n-1)-degree polynomials in terms of the original matrix entries.

6. Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 1 \\ 6 & 3 & 8 \\ 8 & 7 & 10 \end{pmatrix}$$

Compute, by hand,  $\det A$  and  $\det B$ . Likewise for perm A and perm B.

 $\det A = 1$ ,  $\det B = 0$ , perm A = 13, perm B = 400.

7. Cf. previous problem. Compute the following expression involving the standard 3-dimensional inner and outer product of vectors (cf. the columns of B), and provide a geometrical interpretation of your result for det B:

$$\left[ \left( \begin{array}{c} 1\\6\\8 \end{array} \right) \times \left( \begin{array}{c} 2\\3\\7 \end{array} \right) \right] \cdot \left( \begin{array}{c} 1\\8\\10 \end{array} \right)$$

This expression yields 0, indicating that the third column vector of B lies within the 2-dimensional plane spanned by the first two column vectors. This explains why det B = 0.

8. Cf. previous problem. Compute the cofactor and adjugate matrices  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{A}^{T}$ , and  $\tilde{B}^{T}$ .

$$\tilde{A} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} -26 & 4 & 18 \\ -13 & 2 & 9 \\ 13 & -2 & -9 \end{pmatrix}, \quad \tilde{A}^T = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}, \quad \tilde{B}^T = \begin{pmatrix} -26 & -13 & 13 \\ 4 & 2 & -2 \\ 18 & 9 & -9 \end{pmatrix}$$

9. Cf. previous problem. Compute the inverse matrices  $A^{-1}$  and  $B^{-1}$ , if these exist.

Since det A = 1 we have  $A^{-1} = \frac{1}{\det A} \tilde{A}^T = \tilde{A}^T$ , cf. previous problem. Since det B = 0 the matrix  $B^{-1}$  does not exist.

**10.** Cf. previous problem. Compute  $A\tilde{A}^{T}$  and  $B\tilde{B}^{T}$ .

We have  $A\tilde{A}^T = \det AI = I$ , i.e. the 2×2 identity matrix, respectively  $B\tilde{B}^T = \det BI = 0$ , i.e. the 3×3 null matrix.

**11.** Given det A, det B for general  $n \times n$  matrices A, B. What is det  $\tilde{A}$ , det  $A^{T}$ , det $(\lambda A)$ , det(AB), and det  $A^{k}$  for  $\lambda \in \mathbb{R}, k \in \mathbb{Z}$ ?

The following identities hold: det  $\tilde{A} = (\det A)^{n-1}$ , det  $A^T = \det A$ ,  $\det(\lambda A) = \lambda^n \det A$ ,  $\det A^k = (\det A)^k$ .

**12.** Consider the collection  $A_{i_1...i_n}$  for all  $i_1, ..., i_n = 1, ..., n$ . Suppose  $A_{i_1...i_k...i_\ell...i_n} = -A_{i_1...i_\ell...i_k...i_n}$  for any  $1 \le k < \ell \le n$  (complete antisymmetry). Show that  $A_{i_1...i_n} \propto [i_1...i_n]$ .

From complete antisymmetry under each index exchange it follows that all symbols with two or more equal index labels (say  $i_k = i_\ell$ ) must necessarily vanish (as 0 is the only number equal to its opposite). This implies that for nontrivial symbols  $A_{i_1...i_n}$  the index set  $(i_1, ..., i_n)$  must be a permutation of (1, ..., n). Setting  $A_{1...n} = C$  for some constant C, all  $A_{i_1...i_n}$  can be obtained as  $\pm C$  through repetitive index exchanges exploiting the antisymmetry property, viz.  $A_{i_1...i_n} = C[i_1...i_n]$ . Thus the underlying "hypermatrix" A, comprising all  $n^n$  entries  $A_{i_1...i_n}$ , has in fact only a single degree of freedom. **13.** Prove the following identities for the completely antisymmetric symbol in n = 3:

$$[i, j, k] [\ell, m, n] = \delta_{i\ell} \delta_{jm} \delta_{kn} + \delta_{im} \delta_{jn} \delta_{k\ell} + \delta_{in} \delta_{j\ell} \delta_{km} - \delta_{im} \delta_{j\ell} \delta_{kn} - \delta_{i\ell} \delta_{jn} \delta_{km} - \delta_{in} \delta_{jm} \delta_{k\ell}$$

$$\sum_{i=1}^{3} [i, j, k] [i, m, n] = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$$

$$\sum_{i, j=1}^{3} [i, j, k] [i, j, n] = 2\delta_{kn}$$

$$\sum_{i, j, k=1}^{3} [i, j, k] [i, j, k] = 6$$

The first identity follows from the following observations. The l.h.s. is notrivial iff the three entries of each triple (i, j, k) and  $(\ell, m, n)$ , with  $i, j, k, \ell, m, n = 1, 2, 3$ , are all distinct, and therefore equal to some permutation of (1, 2, 3). This implies that the entries of (i, j, k) should pairwise match those of  $(\ell, m, n)$ . The six possible matches are reflected in the Kronecker symbols on the r.h.s. The sign in front of each term expresses whether one needs an even (+) or odd number (-) of reorderings of pairs in  $(\ell, m, n)$  in order to bring these three indices in the same ordering as their matching indices in (i, j, k). Note that once this reordering has been carried out, the product is invariably 1, explaining the amplitude  $\pm 1$  of each term.

Subsequent expressions follow by conducting the indicated index contractions, using  $\sum_{i=1}^{3} \delta_{ii} = 3$  (in 3 dimensions) and  $\sum_{i=1}^{3} \delta_{ij} \delta_{ik} = \delta_{jk}$ .

\* 14. Compute the Gaussian integral  $\gamma(A) = \int_{\mathbb{R}^n} \exp(-x^i A_{ij} x^j) dx$  for a symmetric positive definite  $n \times n$  matrix A. Hint: There exists a rotation matrix R such that  $R^T A R = \Delta$ , in which  $\Delta = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ , with all  $\lambda_p > 0, p = 1, \ldots, n$ .

A symmetric matrix can be diagonalized via a rotation. Thus consider the following substitution of variables: x = Ry, whence  $x^T = y^T R^T$  (note that  $R^T = R^{-1}$ ) and  $dx = \det R \, dy = dy$ . Define the diagonal matrix  $\Delta = R^T A R = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ , with  $\lambda_p > 0$  for all  $p = 1, \ldots, n$  by virtue of positive definiteness. Then, with parameters  $\lambda_p$  considered implicitly as a function of matrix A as indicated,

$$\gamma(A) = \prod_{p=1}^n \int_{\mathbb{R}} \exp(-\lambda_p(y^p)^2) \, dy^p \, .$$

If desired, one may absorb the eigenvalues  $\lambda_p$  into the integration variables by another substitution of variables in order to further simplify the result, viz. consider the rescaling  $z^p = \sqrt{\lambda_p} y^p$ . Then, again assuming  $\lambda_p$  to be given as functions of A,

$$\gamma(A) = \prod_{p=1}^n \frac{1}{\sqrt{\lambda_p}} \int_{\mathbb{R}} \exp(-(z^p)^2) \, dz^p \, .$$

This reduces the problem to a standard one-dimensional Gaussian integral:  $\int_{\mathbb{R}} \exp(-\xi^2) d\xi = \sqrt{\pi}$ . Note that  $\prod_{p=1}^n \lambda_p = \det A$ , so that we finally obtain

$$\gamma(A) = \frac{\sqrt{\pi}^n}{\sqrt{\det A}}$$

\* 15. Compute the extended Gaussian integral  $\gamma(A, s) = \int_{\mathbb{R}^n} \exp(-x^i A_{ij} x^j + s_k x^k) dx$  for a symmetric positive definite  $n \times n$  matrix A and arbitrary "source"  $s \in \mathbb{R}^n$ .

Rewrite the exponent as a quadratic form in x - a for some vector  $a \in \mathbb{R}^n$  and an x-independent (a-dependent) remainder, and solve for a in terms of the source s and the matrix A. In this way one obtains

$$\exp(-x^{i}A_{ij}x^{j} + s_{k}x^{k}) = \exp\left(-(x^{i} - \frac{1}{2}s_{k}B^{ki})A_{ij}(x^{j} - \frac{1}{2}s_{\ell}B^{\ell j})\right)\exp\left(\frac{1}{4}s_{p}B^{pq}s_{q}\right)$$

in which  $B = A^{-1}$ , i.e.  $A_{ij}B^{jk} = \delta_i^k$ . With the help of a substitution of variables,  $y^i = x^i - \frac{1}{2}s_kB^{ki}$ , for which dy = dx, we then obtain

$$\gamma(A,s) = \exp\left(\frac{1}{4}s_p B^{pq} s_q\right) \int_{\mathbb{R}^n} \exp\left(-y^i A_{ij} y^j\right) dy = \frac{\sqrt{\pi}^n}{\sqrt{\det A}} \exp\left(\frac{1}{4}s_p B^{pq} s_q\right) = \frac{\sqrt{\pi}^n}{\sqrt{\det A}} \exp\left(\frac{1}{4}s^T A^{-1}s\right) \,.$$

\* 16. Cf. previous problem. Consider the following integral:  $\gamma^{i_1...i_p}(A) = \int_{\mathbb{R}^n} x^{i_1} \dots x^{i_p} \exp(-x^i A_{ij} x^j) dx$ . Express  $\gamma^{i_1...i_p}(A)$  in terms of  $\gamma(A, s)$ . (You don't need to compute  $\gamma^{i_1...i_p}(A)$  explicitly.)

Observe that

$$\gamma^{i_1 \dots i_p}(A) = \left. \partial^{i_1} \dots \partial^{i_p} \gamma(A, s) \right|_{s=0} \,,$$

in which  $\partial^i = \partial/\partial s_i$ . Together with the expression obtained for  $\gamma(A, s)$  in the previous problem this admits a closed-form solution.

### 2. Tensor Calculus

**1.** Expand in n = 2 and n = 3 dimensions, respectively:  $X_i Y^{ij} X_j$ . Argue why we may assume, without loss of generality, that  $Y^{ij} = Y^{ji}$  in this expression ("automatic" symmetry).

In the following expressions all superscripts are to be interpreted as contravariant indices (labels), except if attached to a parenthesized expression, in which case they denote powers, i.c. squares:

- for n = 2:  $X_i Y^{ij} X_j = Y^{11} (X_1)^2 + 2Y^{12} X_1 X_2 + Y^{22} (X_2)^2$ ;
- for n = 3:  $X_i Y^{ij} X_j = Y^{11} (X_1)^2 + 2Y^{12} X_1 X_2 + 2Y^{13} X_1 X_3 + Y^{22} (X_2)^2 + 2Y^{23} X_2 X_3 + Y^{33} (X_3)^2$ .

Note that we have used symmetry to simplify the expressions by noting that  $Y^{21} = Y^{12}$ ,  $Y^{31} = Y^{13}$ ,  $Y^{32} = Y^{23}$ , i.e. the effective indices of  $Y^{ij}$  may be rearranged such that  $i \leq j$  provided one takes into account a proper combinatorial factor, i.e. a factor 2 for the mixed terms. The legitimacy of the symmetry assumption  $Y^{ij} = Y^{ji}$  can be seen as follows. In general one may write any square matrix as a sum of a symmetric and an antisymmetric part:  $Y = Y_s + Y_a$ , with entries

$$Y^{ij}_{\mathrm{s}} = rac{1}{2} \left(Y^{ij} + Y^{ji}
ight) \quad \mathrm{and} \quad Y^{ij}_{\mathrm{a}} = rac{1}{2} \left(Y^{ij} - Y^{ji}
ight) \;.$$

As a result,  $X_i Y^{ij} X_j = X_i (Y_s^{ij} + Y_a^{ij}) X_j = X_i Y_s^{ij} X_j + X_i Y_a^{ij} X_j = X_i Y_s^{ij} X_j$ . The latter step follows from  $X_i Y_a^{ij} X_j \stackrel{*}{=} X_j Y_a^{ji} X_i \stackrel{*}{=} -X_j Y_a^{ij} X_i = -X_i Y_a^{ij} X_j$ . In step  $\star$  we have merely relabelled dummies, in step  $\star$  we have used antisymmetry  $Y_a^{ji} = -Y_a^{ij}$ , the final step is trivial. Since the antisymmetric part of Y does not contribute to the result we may as well assume Y to be symmetric from the outset, as done in the above.

**2.** Expand in n = 2 and n = 3 dimensions, respectively:  $X_{ij}Y^{ij}$ . Are we allowed to assume that  $X_{ij}$  or  $Y^{ij}$  is symmetric in this expression?

We now have (again all superscripts are labels, not powers)

- for n = 2:  $X_{ij}Y^{ij} = X_{11}Y^{11} + X_{12}Y^{12} + X_{21}Y^{21} + X_{22}Y^{22}$ ;
- for n = 3:  $X_{ij}Y^{ij} = X_{11}Y^{11} + X_{12}Y^{12} + X_{13}Y^{13} + X_{21}Y^{21} + X_{22}Y^{22} + X_{23}Y^{23} + X_{31}Y^{31} + X_{32}Y^{32} + X_{33}Y^{33}$ .

There is no reason why there should be any dependencies, such as symmetries, among the  $X_{ij}$  or  $Y^{ij}$ .

**3.** Show that 
$$\delta_i^i = n$$
 and  $\delta_k^i \delta_j^k = \delta_j^i$ .

We have  $\delta_i^i = \delta_1^1 + \ldots + \delta_n^n = \underbrace{1 + \ldots + 1}_{n \text{ terms}} = n$ . Furthermore,  $\delta_k^i \delta_j^k \equiv \sum_{k=1}^n \delta_k^i \delta_j^k = \delta_j^i$ , because the first Kronecker symbol in the sum,  $\delta_k^i$ , singles

out only the term for which k = i, in which case it evaluates to 1, leaving for the second Kronecker symbol  $\delta_i^{k=i}$  as the only effective term.

4. Show that  $\delta^i_i X^j_i = X^i_i$ .

Of all the *j*-labelled terms in the double sum  $\delta_j^i X_i^j \equiv \sum_{i=1}^n \sum_{j=1}^n \delta_j^i X_i^j$  the only ones that are effective due to the  $\delta_j^i$  factor are those terms with j = i, leaving  $X_i^{j=i} \equiv \sum_{i=1}^n X_i^i$ , i.e. *n* instead of the  $n^2$  terms that are a priori present in the full  $\sum_{i=1}^n \sum_{j=1}^n$ -sum.

**5.** Suppose  $x^i = A^i_j y^j$  and  $y^i = B^i_j x^j$  for all  $y \in \mathbb{R}^n$ . Prove that  $A^i_k B^k_j = \delta^i_j$ .

Multiply the first equation with  $B_i^k$  and sum over i = 1, ..., n. Using the second identity we then obtain  $y^k = B_i^k x^i = B_i^k A_j^i y^j$  for all  $y^k, k = 1, ..., n$ , which implies that  $B_i^k A_j^i = \delta_j^k$ . Alternatively you can see this by differentiating  $y^k = B_i^k A_j^i y^j$  w.r.t.  $y^j$ , yielding  $\delta_j^k = B_i^k A_j^i$ .

**6.** Consider a Cartesian basis  $\{\partial_1 \equiv \partial_x, \partial_2 \equiv \partial_y\}$  spanning a 2-dimensional plane. (Here  $\partial_i$  is shorthand for  $\partial/\partial x^i$  if  $x^i$  denotes the *i*-th Cartesian coordinate; in  $\mathbb{R}^2$  we identify  $x^1 \equiv x, x^2 \equiv y$ .) Let  $\overline{x}^i$  denote the *i*-th polar coordinate, with  $\overline{x}^1 \equiv r$  (radial distance) and  $\overline{x}^2 \equiv \phi$  (polar angle), such that  $x = r \cos \phi, y = r \sin \phi$ . Assume  $\overline{\partial}_j = A_j^i \partial_i$  (with  $\overline{\partial}_1 \equiv \partial_r, \overline{\partial}_2 \equiv \partial_{\phi}$ ). Argue why this assumption holds, and compute the matrix A.

The assumption holds by virtue of the chain rule, viz.  $A_j^i = \frac{\partial x^i}{\partial \overline{x}^j}$ . Taking *i* and *j* as row, respectively column index, we have

$$A = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos \phi & -r \sin \phi \\ \sin \phi & r \cos \phi \end{pmatrix}.$$

7. Cf. previous problem, but now for the relation between Cartesian and spherical coordinates in  $\mathbb{R}^3$ , defined by  $x = r \sin \theta \cos \phi$ ,  $y = r \sin \theta \sin \phi$ ,  $z = r \cos \theta$ , with radial distance  $\overline{x}^1 \equiv r$ , polar angle  $\overline{x}^2 \equiv \theta$  and azimuthal angle  $\overline{x}^3 \equiv \phi$ , and, correspondingly,  $\overline{\partial}_1 \equiv \partial_r$ ,  $\overline{\partial}_2 \equiv \partial_\theta$ ,  $\overline{\partial}_3 \equiv \partial_\phi$ .

The assumption holds by virtue of the chain rule, viz.  $A_j^i = \frac{\partial x^i}{\partial x^j}$ . Taking *i* and *j* as row, respectively column index, we have

$$A = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}.$$

8. Cf. previous two problems. Consider the dual Cartesian bases  $\{dx^1 \equiv dx, dx^2 \equiv dy\}$  in two dimensions, respectively  $\{dx^1 \equiv dx, dx^2 \equiv dy, dx^3 \equiv dz\}$  in three dimensions. Assume  $dx^i = C_j^i d\overline{x}^j$ , with dual bases  $\{d\overline{x}^1 \equiv dr, d\overline{x}^2 \equiv d\phi\}$ , respectively  $\{d\overline{x}^1 \equiv dr, d\overline{x}^2 \equiv d\theta, d\overline{x}^3 \equiv d\phi\}$ . Show that C = A, i.e. the same matrix as computed in the previous two problems. (Notice the difference!)

By virtue of the chain rule for differentials we have  $dx^i = C^i_j d\overline{x}^j$ , with  $C^i_j = \frac{\partial x^i}{\partial \overline{x}^j}$ , i.e. the same matrix C = A as evaluated in the previous problem. Whereas in the previous problem the *new* derivatives  $\overline{\partial}_i$  were expressed as linear combinations of the *old* ones  $\partial_j$ , we now have the *old* differentials  $dx^i$  expressed as linear combinations of the *new* ones  $d\overline{x}^j$ , with the *same* coefficients.

9. Let  $\mathbf{x} = x^i \mathbf{e}_i \in V$ . Show that the map  $\hat{\mathbf{a}} : V \to \mathbb{R}$  defined by  $\hat{\mathbf{a}}(\mathbf{x}) = a_i x^i$  is a linear operator.

For  $\mathbf{x}, \mathbf{y} \in V, \lambda, \mu \in \mathbb{R}$ , we have  $\hat{\mathbf{a}}(\lambda \mathbf{x} + \mu \mathbf{y}) = a_i(\lambda x^i + \mu y^i) = \lambda a_i x^i + \mu a_i y^i = \lambda \hat{\mathbf{a}}(\mathbf{x}) + \mu \hat{\mathbf{a}}(\mathbf{y})$ .

10. Cf. previous problem. A linear operator of this type is known as a *covector*, notation  $\hat{\mathbf{a}} \in V^* \equiv \mathscr{L}(V, \mathbb{R})$ . Explain why a covector  $\hat{\mathbf{a}}$  (respectively covector space  $V^*$ ) is formally a vector (respectively vector space).

For  $\lambda, \mu \in \mathbb{R}, \hat{\mathbf{a}}, \hat{\mathbf{b}} \in V^*$ , and arbitrary  $\mathbf{x} \in V$ , we have, by definition,  $(\lambda \hat{\mathbf{a}} + \mu \hat{\mathbf{b}})(\mathbf{x}) = \lambda \hat{\mathbf{a}}(\mathbf{x}) + \mu \hat{\mathbf{b}}(\mathbf{x})$ . With this definition,  $V^*$  acquires a vector space structure, and its elements, i.c.  $\hat{\mathbf{a}}, \hat{\mathbf{b}} \in V^*$ , are, by definition, vectors.

11. Consider  $\hat{\mathbf{a}} \in V^*$  with prototype  $\hat{\mathbf{a}} : V \to \mathbb{R} : \mathbf{x} \mapsto \hat{\mathbf{a}}(\mathbf{x}) = a_i x^i$ . Argue why this naturally provides an alternative interpretation of  $\mathbf{x} \in V$  as an element of  $V^{**} = (V^*)^* \sim V$ .

Notice the symmetric role played by the coefficients  $a_i$  of the linear map  $\hat{\mathbf{a}} \in V^*$  and its arguments  $x^i$ . We might as well call  $x^i$  the coefficients, and  $a_i$  the arguments, if we reinterpret the linear map as  $\mathbf{x} \in V^{**}$ , i.e.  $\mathbf{x} : V^* \to \mathbb{R} : \hat{\mathbf{a}} \mapsto a_i x^i$ , i.e.  $\mathbf{x}(\hat{\mathbf{a}}) \stackrel{\text{def}}{=} \hat{\mathbf{a}}(\mathbf{x})$ .

12. Suppose  $\langle \hat{\omega}, \mathbf{v} \rangle = C_j^i \omega_i v^j = \overline{C}_j^i \overline{\omega}_i \overline{v}^j$  relative to dual bases  $\{\mathbf{e}_i, \hat{\mathbf{e}}^i\}$  (middle term), respectively  $\{\mathbf{f}_i, \hat{\mathbf{f}}^i\}$  (right hand side), in which  $C_j^i$  and  $\overline{C}_j^i$  are coefficients ("holors") to be determined. Show that the holor is basis independent, i.e.  $C_j^i = \overline{C}_j^i$ , and compute its components explicitly.

Inserting the decompositions  $\hat{\omega} = \omega_i \hat{\mathbf{e}}^i$  and  $\mathbf{v} = v^j \mathbf{e}_j$  into the Kronecker tensor yields, using bilinearity and duality,  $\langle \hat{\omega}, \mathbf{v} \rangle = \omega_i v^j \langle \hat{\mathbf{e}}^i, \mathbf{e}_j \rangle = \omega_i v^j \delta_j^i$ , so that apparently  $C_j^i = \delta_j^i$ . This is basis independent, so by the same token we have  $\overline{C}_j^i = \delta_j^i$ .

**13.** Suppose  $\{\mathbf{e}_i\}$  and  $\{\hat{\mathbf{e}}^i\}$  are dual bases of V, respectively  $V^*$ , and  $\mathbf{e}_i = A_i^j \mathbf{f}_j$ ,  $\hat{\mathbf{e}}^i = B_j^i \hat{\mathbf{f}}^j$  for some (a priori unrelated) transformation matrices A and B. Show that if  $\{\mathbf{f}_i, \hat{\mathbf{f}}^j\}$  constitute dual bases then  $B_k^j A_i^k = \delta_i^j$ .

By virtue of duality we have  $\delta_i^j = \langle \hat{\mathbf{e}}^j, \mathbf{e}_i \rangle = \langle B_k^j \hat{\mathbf{f}}^k, A_i^\ell \mathbf{f}_\ell \rangle = B_k^j A_i^\ell \langle \hat{\mathbf{f}}^k, \mathbf{f}_\ell \rangle \stackrel{*}{=} B_k^j A_i^\ell \delta_\ell^k = B_k^j A_i^\ell \delta_\ell^k$ . The equality marked by \* follows if  $\{\mathbf{f}_j, \hat{\mathbf{f}}^i\}$  define dual bases, i.e. if  $\langle \hat{\mathbf{f}}^i, \mathbf{f}_j \rangle = \delta_j^i$ .

\* 14. Show that the following "length functional"  $\mathscr{L}(\gamma)$  for a parameterized curve  $\gamma : [T_-, T_+] \to \mathbb{R}^n : t \mapsto \gamma(t)$  with fixed end points  $X_{\pm} = \gamma(T_{\pm})$  is independent of the parametrization:

$$\mathscr{L}(\gamma) = \int_{T_-}^{T_+} \sqrt{g_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t)} \, dt \, .$$

Here  $\dot{\gamma}(t) = \dot{\gamma}^i(t)\mathbf{e}_i$  denotes the derivative of the curve, expanded relative to a fiducial basis  $\{\mathbf{e}_i\}$ , and  $\gamma_{ij}(x)$  are the components of the inner product defined at base point  $x \in \mathbb{R}^n$ . To this end, consider a reparametrization of the form s = s(t), with  $\dot{s}(t) > 0$ , say.

Substitution of variables, with s = s(t) or, equivalently, t = t(s), setting  $\gamma(t) = \xi(s(t))$  and defining  $\dot{\xi}(s) = d\xi(s)/ds$ , yields, by the chain rule,

$$\mathscr{L}(\gamma) = \int_{T_{-}}^{T_{+}} \sqrt{g_{ij}(\gamma(t)) \,\dot{\gamma}^{i}(t) \,\dot{\gamma}^{j}(t)} \, dt = \int_{T_{-}}^{T_{+}} \sqrt{g_{ij}(\xi(s(t))) \,\dot{\xi}^{i}(s(t)) \,\dot{s}(t) \,\dot{\xi}^{j}(s(t)) \,\dot{s}(t)} \, dt = \int_{S_{-}}^{S_{+}} \sqrt{g_{ij}(\xi(s)) \,\dot{\xi}^{i}(s) \,\dot{\xi}^{j}(s)} \, ds \,,$$

in which we have used  $ds = \dot{s}(t)dt$ , and in which  $S_{\pm} = s(T_{\pm})$  denote the new parameter values of the (fixed) end points  $X_{\pm}$ .

**15.** Cf. previous problem. The parameter *s* is called an *affine parameter* if, along the entire parameterized curve  $\xi : [S_-, S_+] \to \mathbb{R}^n : s \mapsto \xi(s)$ , we have  $\|\dot{\xi}(s)\| = 1$  ("unit speed parameterization"), or  $(\dot{\xi}(s)|\dot{\xi}(s)) = 1$ , in which the l.h.s. pertains to the inner product at point  $\xi(s) \in \mathbb{R}^n$ . In other words,  $g_{ij}(\xi(s))\dot{\xi}^i(s)\dot{\xi}^j(s) = 1$ . Show that this can always be realized through suitable reparameterization starting from an *arbitrarily* parameterized curve  $\gamma : [T_-, T_+] \to \mathbb{R}^n : t \mapsto \gamma(t)$ .

Cf. previous problem for notation. Note that  $(\dot{\gamma}(t)|\dot{\gamma}(t)) = f^2(t)$  for some positive definite function  $f : [T_-, T_+] \to \mathbb{R} : t \mapsto f(t)$ . By the chain rule we have  $f^2(t) = (\dot{\xi}(s(t))\dot{s}(t)|\dot{\xi}(s(t))\dot{s}(t)) = (\dot{\xi}(s(t))|\dot{\xi}(s(t)))\dot{s}^2(t)$ . If we now choose the reparameterization such that  $\dot{s}(t) = f(t)$  with arbitrarily fixed initial condition  $s(T_-) = S_-$  (admitting a unique solution), then we get  $(\dot{\xi}(s)|\dot{\xi}(s)) = 1$  along the entire reparameterized curve  $\xi : [S_-, S_+] \to \mathbb{R}^n : s \mapsto \xi(s)$ . Note that the reparameterization is one-to-one.

16. The figure below shows a pictorial representation of vectors  $(\mathbf{v}, \mathbf{w} \in V)$  and covectors  $(\boldsymbol{\omega} \in V^*)$  in terms of graphical primitives. In this picture, a vector is denoted by a directed arrow, and a covector by an equally spaced set of level lines along a directed normal (i.e. "phase" increases in the direction of the directed normal, attaining consecutive integer values on the level lines drawn in the figure). Give a graphical interpretation of the contraction  $\langle \boldsymbol{\omega}, \mathbf{v} \rangle \in \mathbb{R}$ , and estimate from the figure the values of  $\langle \boldsymbol{\omega}, \mathbf{v} \rangle$ ,  $\langle \boldsymbol{\omega}, \mathbf{w} \rangle$ , and  $\langle \boldsymbol{\omega}, \mathbf{v} + \mathbf{w} \rangle$ . Are these values consistent with the linearity of  $\langle \boldsymbol{\omega}, \cdot \rangle$ ?



The contraction  $\langle \boldsymbol{\omega}, \mathbf{v} \rangle$  equals the (fractional) number of level lines of  $\boldsymbol{\omega}$  intersected by the arrow  $\mathbf{v}$ , with a sign depending on whether the progression along the level lines is in forward (+) or backward (-) direction, in other words, the phase difference between tip and tail of the arrow relative to the "planar wave" pattern. From the picture we estimate  $\langle \boldsymbol{\omega}, \mathbf{v} \rangle \approx 3$ ,  $\langle \boldsymbol{\omega}, \mathbf{w} \rangle \approx 2$ , and  $\langle \boldsymbol{\omega}, \mathbf{v} + \mathbf{w} \rangle \approx 5$ , consistent with the linearity constraint  $\langle \boldsymbol{\omega}, \mathbf{v} \rangle + \langle \boldsymbol{\omega}, \mathbf{w} \rangle = \langle \boldsymbol{\omega}, \mathbf{v} + \mathbf{w} \rangle$ .

17. An inner product on V induces an inner product on V<sup>\*</sup>. Recall that for  $\mathbf{x} = x^i \mathbf{e}_i \in V$  and  $\mathbf{y} = y^j \mathbf{e}_j \in V$  we have  $(\mathbf{x}|\mathbf{y}) = g_{ij}x^iy^j$ . Let  $\hat{\mathbf{x}} = x_i\hat{\mathbf{e}}^i \in V^*$ ,  $\hat{\mathbf{y}} = y_j\hat{\mathbf{e}}^j \in V^*$ , with  $\langle \hat{\mathbf{e}}^i, \mathbf{e}_j \rangle = \delta_j^i$ . Define  $(\hat{\mathbf{x}}|\hat{\mathbf{y}})_* = (\flat \hat{\mathbf{x}}|\flat \hat{\mathbf{y}})$ . Show that  $(\hat{\mathbf{x}}|\hat{\mathbf{y}})_* = g^{ij}x_iy_j$ , in which  $g^{ik}g_{kj} = \delta_j^i$ .

We have  $(\hat{\mathbf{x}}|\hat{\mathbf{y}})_* = (\flat \hat{\mathbf{x}}|\flat \hat{\mathbf{y}}) = \left(g^{ik}x_i\mathbf{e}_k|g^{j\ell}y_\ell\mathbf{e}_j\right) = g^{ik}g^{j\ell}x_iy_\ell\left(\mathbf{e}_k|\mathbf{e}_j\right) = g^{ik}g_{kj}g^{j\ell}x_iy_\ell = g^{ij}x_iy_j.$ 

**18.** Let  $\hat{\mathbf{u}}, \hat{\mathbf{v}} \in V^*$ . Prove equivalence of nilpotency and antisymmetry of the wedge product, i.e.  $\hat{\mathbf{u}} \wedge \hat{\mathbf{u}} = 0$  iff  $\hat{\mathbf{u}} \wedge \hat{\mathbf{v}} = -\hat{\mathbf{v}} \wedge \hat{\mathbf{u}}$ .

Starting from nilpotency, consider  $0 = (\hat{\mathbf{u}} + \hat{\mathbf{v}}) \land (\hat{\mathbf{u}} + \hat{\mathbf{v}}) = \hat{\mathbf{u}} \land \hat{\mathbf{v}} + \hat{\mathbf{v}} \land \hat{\mathbf{u}}$ , in which the last step follows from bilinearity and nilpotency of the  $\land$ -product. Vice versa, antisymmetry implies  $\hat{\mathbf{u}} \land \hat{\mathbf{u}} = -\hat{\mathbf{u}} \land \hat{\mathbf{u}}$ , whence  $\hat{\mathbf{u}} \land \hat{\mathbf{u}} = 0$ .

**19.** Let  $\pi, \pi' : \{1, \ldots, p\} \to \{1, \ldots, p\} : k \mapsto \pi(k)$  be two bijections. By abuse of notation these can be identified with permutations on any symbolic set  $\Omega_p = \{(a_1, \ldots, a_p)\}$  consisting of *p*-tuples of labeled symbols:  $\pi : \Omega_p \to \Omega_p : (a_1, \ldots, a_p) \mapsto (a_{\pi(1)}, \ldots, a_{\pi(p)})$ , and likewise for  $\pi'$ . Let  $\pi(1, \ldots, p) = (\pi(1), \ldots, \pi(p))$ , respectively  $\pi'(1, \ldots, p) = (\pi'(1), \ldots, \pi'(p))$ , argue that  $\pi'' = \pi'\pi$ , i.e. the (right-to-left) concatenation of  $\pi'$  and  $\pi$ , defines another permutation.

Consider  $\pi''(1,\ldots,p) = \pi'\pi(1,\ldots,p) = \pi'(\pi(1),\ldots,\pi(p)) = (\pi'\pi(1),\ldots,\pi'\pi(p))$ . Since  $\pi,\pi': \{1,\ldots,p\} \to \{1,\ldots,p\}$  are bijective, so is their composition  $\pi'' = \pi'\pi: \{1,\ldots,p\} \to \{1,\ldots,p\}$ , whence we must define  $\pi'': \Omega_p \to \Omega_p: (a_1,\ldots,a_p) \mapsto (a_{\pi''(1)},\ldots,a_{\pi''(p)})$ . In particular,  $\pi''(1,\ldots,p) = (\pi''(1),\ldots,\pi''(p))$ , i.e. a permutation of  $(1,\ldots,p)$ .

**20.** Argue that the concatenation  $\pi'' = \pi' \pi$  of two permutations  $\pi'$  and  $\pi$  satisfies sgn  $\pi'' = \text{sgn } \pi' \text{ sgn } \pi$ .

Although the absolute number of pairwise toggles of entries in the *p*-tuple  $\pi(1, \ldots, p) = (\pi(1), \ldots, \pi(p))$  needed to rearrange its entries into the original ordering  $(1, \ldots, p)$  is not unique, it is always either even or odd for any particular permutation. Thus  $\operatorname{sgn} \pi = \pm 1$ , indicating whether this number is even (+) or odd (-), is well-defined. Using " $\pm 1$ " as synonyms for "any even/odd number", rearranging  $\pi''(1, \ldots, p) = (\pi''(1), \ldots, \pi''(p)) = (\pi'\pi(1), \ldots, \pi'\pi(p))$  into  $\pi'(1, \ldots, p) = (\pi'(1), \ldots, \pi'(p))$  requires  $\operatorname{sgn} \pi$  pairwise toggles of entries, while subsequent rearrangement of  $\pi'(1, \ldots, p) = (\pi'(1), \ldots, \pi'(p))$  requires another  $\operatorname{sgn} \pi'$  number. The total number is thus  $\operatorname{sgn} \pi'' = \operatorname{sgn} \pi'$  sgn  $\pi$ .

**21.** Show that the symmetrisation map  $\mathscr{S} : \mathbf{T}_p^0(V) \to \bigvee_p(V)$  is idempotent, i.e.  $\mathscr{S} \circ \mathscr{S} = \mathscr{S}$ .

Consider  $(\mathscr{S} \circ \mathscr{S})(\mathbf{T})(\mathbf{v}_1, \dots, \mathbf{v}_p) = \frac{1}{p!} \sum_{\pi} \mathscr{S}(\mathbf{T})(\mathbf{v}_{\pi(1)}, \dots, \mathbf{v}_{\pi(p)}) = \frac{1}{p!} \sum_{\pi} \frac{1}{p!} \sum_{\pi'} \mathbf{T}(\mathbf{v}_{\pi'\pi(1)}, \dots, \mathbf{v}_{\pi'\pi(p)})$ . Now  $\pi'\pi = \pi''$  is again a permutation of  $(1, \dots, p)$ , so we may rewrite the r.h.s. as  $\frac{1}{p!} \sum_{\pi} \frac{1}{p!} \sum_{\pi''} \mathbf{T}(\mathbf{v}_{\pi''(1)}, \dots, \mathbf{v}_{\pi''(p)})$ . Since the terms in the innermost  $\pi''$ -sum

do not depend on  $\pi$ , the outermost  $\pi$ -sum adds p! identical terms, cancelling one combinatorial factor:  $\sum_{\pi} \frac{1}{p!} = 1$ . Thus  $(\mathscr{S} \circ \mathscr{S})(\mathbf{T})(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \frac{1}{p!} \sum_{\pi''} \mathbf{T}(\mathbf{v}_{\pi''(1)}, \ldots, \mathbf{v}_{\pi''(p)}) = \mathscr{S}(\mathbf{T})(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ . Since this holds for all vector arguments  $\mathbf{v}_k$ ,  $k = 1, \ldots, p$ , we have the operator identity  $\mathscr{S} \circ \mathscr{S} = \mathscr{S}$ .

22. Show that the antisymmetrisation map  $\mathscr{A}: \mathbf{T}_p^0(V) \to \bigwedge_p(V)$  is idempotent, i.e.  $\mathscr{A} \circ \mathscr{A} = \mathscr{A}$ .

Consider  $(\mathscr{A} \circ \mathscr{A})(\mathbf{T})(\mathbf{v}_1, \dots, \mathbf{v}_p) = \frac{1}{p!} \sum_{\pi} \operatorname{sgn} \pi \mathscr{A}(\mathbf{T})(\mathbf{v}_{\pi(1)}, \dots, \mathbf{v}_{\pi(p)}) = \frac{1}{p!} \sum_{\pi} \operatorname{sgn} \pi \frac{1}{p!} \sum_{\pi'} \operatorname{sgn} \pi' \mathbf{T}(\mathbf{v}_{\pi'\pi(1)}, \dots, \mathbf{v}_{\pi'\pi(p)}).$  Now  $\pi'\pi = \pi''$  is again a permutation of  $(1, \dots, p)$ , with  $\operatorname{sgn} \pi'' = \operatorname{sgn} \pi' \operatorname{sgn} \pi$ , so we may rewrite the r.h.s. as  $\frac{1}{p!} \sum_{\pi} \frac{1}{p!} \sum_{\pi''} \operatorname{sgn} \pi'' \mathbf{T}(\mathbf{v}_{\pi''(1)}, \dots, \mathbf{v}_{\pi''(p)}).$ Since the terms in the innermost  $\pi''$ -sum do not depend on  $\pi$ , the outermost  $\pi$ -sum adds p! identical terms, cancelling one combinatorial factor:  $\sum_{\pi} \frac{1}{p!} = 1$ . Thus  $(\mathscr{A} \circ \mathscr{A})(\mathbf{T})(\mathbf{v}_1, \dots, \mathbf{v}_p) = \frac{1}{p!} \sum_{\pi''} \operatorname{sgn} \pi'' \mathbf{T}(\mathbf{v}_{\pi''(1)}, \dots, \mathbf{v}_{\pi''(p)}) = \mathscr{A}(\mathbf{T})(\mathbf{v}_1, \dots, \mathbf{v}_p).$  Since this holds for all vector arguments  $\mathbf{v}_k$ ,  $k = 1, \dots, p$ , we have the operator identity  $\mathscr{A} \circ \mathscr{A} = \mathscr{A}$ .

**23.** Cf. the two previous problems. Show that  $\mathscr{A} \circ \mathscr{S} = \mathscr{S} \circ \mathscr{A} = 0 \in \mathscr{L}(\mathbf{T}_p^0(V), \mathbf{T}_p^0(V))$ , i.e. the null operator on  $\mathbf{T}_p^0(V)$  for  $p \ge 2$ . What if p = 0, 1?

For  $p \ge 2$ , assume  $\mathbf{T} \in \mathscr{A}(\mathbf{T}_p^0(V)) \cap \mathscr{S}(\mathbf{T}_p^0(V)) = \bigwedge_p(V) \cap \bigvee_p(V)$ , i.e.  $\mathbf{T}$  is both symmetric as well as antisymmetric. Then, for  $1 \le k \ne \ell \le p$ ,  $\mathbf{T}(\mathbf{v}_1, \dots, \mathbf{v}_k, \dots, \mathbf{v}_\ell, \dots, \mathbf{v}_p) \stackrel{s}{=} \mathbf{T}(\mathbf{v}_1, \dots, \mathbf{v}_\ell, \dots, \mathbf{v}_k, \dots, \mathbf{v}_p) \stackrel{a}{=} -\mathbf{T}(\mathbf{v}_1, \dots, \mathbf{v}_k, \dots, \mathbf{v}_\ell, \dots, \mathbf{v}_p)$ , in which symmetry (s), respectively antisymmetry (a) has been used in the last two steps. So  $\bigwedge_p(V) \cap \bigvee_p(V) = \{0\}$ , i.e.  $\mathscr{A} \circ \mathscr{S} = \mathscr{S} \circ \mathscr{A} = 0 \in \mathscr{L}(\mathbf{T}_p^0(V), \mathbf{T}_p^0(V))$ . For p = 0, 1 the maps  $\mathscr{A}$  and  $\mathscr{S}$  act trivially:  $\mathscr{A} = \mathscr{S} = \mathrm{id} : \mathbf{T}_{p=0,1}^0(V) \to \mathbf{T}_{p=0,1}^0(V)$ .

24. Let  $t_{i_1...i_p}$  be the holor of  $\mathbf{T} \in \mathbf{T}_p^0(V)$ . Show that the holor of  $\mathscr{S}(\mathbf{T}) \in \bigwedge_p(V)$  is  $t_{(i_1...i_p)} \stackrel{\text{def}}{=} \frac{1}{p!} \sum_{\pi} t_{\pi(i_1)...\pi(i_p)}$ .

We have

$$\begin{split} \mathscr{S}(\mathbf{T})(\mathbf{v}_{1},\ldots,\mathbf{v}_{p}) &= \\ & \frac{1}{p!}\sum_{\pi}\mathbf{T}(\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(p)}) = \frac{1}{p!}\sum_{\pi}t_{i_{1}\ldots i_{p}}v_{\pi(1)}^{i_{1}}\ldots v_{\pi(p)}^{i_{p}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}t_{i_{\pi}(1)}\ldots i_{\pi(p)}v_{\pi(1)}^{i_{\pi(1)}}\ldots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}t_{i_{\pi(1)}\ldots i_{\pi(p)}}v_{\pi(1)}^{i_{1}}\ldots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}t_{i_{\pi(1)}\ldots i_{\pi(p)}}v_{\pi(1)}^{i_{1}}\ldots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}t_{i_{\pi(1)}\ldots i_{\pi(p)}}v_{\pi(1)}^{i_{1}}\ldots v_{\pi(p)}^{i_{p}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}t_{i_{\pi(1)}}\ldots v_{\pi(p)}^{i_{\pi(1)}}\cdots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}t_{i_{\pi(1)}}\cdots v_{\pi(p)}^{i_{\pi(1)}}\cdots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}t_{i_{\pi(1)}}\cdots v_{\pi(p)}^{i_{\pi(1)}}\cdots v_{\pi(p)}^{i_{\pi(1)}}\cdots v_{\pi(p)}^{i_{\pi(1)}}\cdots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}t_{i_{\pi(1)}}\cdots v_{\pi(p)}^{i_{\pi(1)}}\cdots v_{\pi(p)}^{i_{\pi(p)}}$$

In step \* we have relabeled dummy summation indices in each of the p! terms of the  $\pi$ -sum, while step \* relies on a trivial reordering of commuting factors.

**25.** Cf. previous problem. Show that the holor of  $\mathscr{A}(\mathbf{T}) \in \bigwedge_p(V)$  is  $t_{[i_1...i_p]} \stackrel{\text{def}}{=} \frac{1}{p!} \sum_{\pi} \operatorname{sgn} \pi t_{\pi(i_1)...\pi(i_p)}$ .

We have

$$\begin{aligned} \mathscr{A}(\mathbf{T})(\mathbf{v}_{1},\ldots,\mathbf{v}_{p}) &= \\ & \frac{1}{p!}\sum_{\pi}\operatorname{sgn}\pi\,\mathbf{T}(\mathbf{v}_{\pi(1)},\ldots,\mathbf{v}_{\pi(p)}) = \frac{1}{p!}\sum_{\pi}\operatorname{sgn}\pi\,t_{i_{1}\ldots i_{p}}v_{\pi(1)}^{i_{1}}\ldots v_{\pi(p)}^{i_{p}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}\operatorname{sgn}\pi\,t_{i_{\pi(1)}\ldots i_{\pi(p)}}v_{\pi(1)}^{i_{\pi(1)}}\ldots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}\operatorname{sgn}\pi\,t_{i_{\pi(1)}\ldots i_{\pi(p)}}v_{\pi(1)}^{i_{\pi(1)}}\ldots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}\operatorname{sgn}\pi\,t_{i_{\pi(1)}\ldots i_{\pi(p)}}v_{\pi(1)}^{i_{1}}\ldots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}\operatorname{sgn}\pi\,t_{i_{\pi(1)}\ldots i_{\pi(p)}}v_{\pi(1)}^{i_{\pi(1)}}\ldots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}\operatorname{sgn}\pi\,t_{i_{\pi(1)}\ldots i_{\pi(p)}}v_{\pi(1)}^{i_{\pi(p)}}\ldots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}\operatorname{sgn}\pi\,t_{i_{\pi(1)}\ldots i_{\pi(p)}}v_{\pi(1)}^{i_{\pi(p)}}\ldots v_{\pi(p)}^{i_{\pi(p)}} \stackrel{*}{=} \frac{1}{p!}\sum_{\pi}\operatorname{sgn}\pi\,t_{i_{\pi(1)}\ldots i_{\pi(p)}}v_{\pi(p)}^{i_{\pi(1)}}\ldots v_{\pi(p)}^{i_{\pi(p)}}\cdots v_{\pi(p)}^{i_{\pi(p)}}$$

In step \* we have relabeled dummy summation indices in each of the p! terms of the  $\pi$ -sum, while step \* relies on a trivial reordering of commuting factors.

**26.** Write out the symmetrised holor  $t_{(ijk)}$  in terms of  $t_{ijk}$ . Likewise for the antisymmetrised holor  $t_{[ijk]}$ .

We have  $t_{(ijk)} = \frac{1}{3!} \sum_{\pi} t_{\pi(i)\pi(j)\pi(k)} = \frac{1}{6} (t_{123} + t_{231} + t_{312} + t_{213} + t_{132} + t_{321})$ , respectively  $t_{[ijk]} = \frac{1}{3!} \sum_{\pi} \operatorname{sgn} \pi t_{\pi(i)\pi(j)\pi(k)} = \frac{1}{6} (t_{123} + t_{231} + t_{312} - t_{213} - t_{132} - t_{321})$ .

27. Simplify the expressions of the previous problem as far as possible if  $t_{ijk}$  is known to possess the partial symmetry property  $t_{ijk} = t_{jik}$ .

In this case we encounter identical pairs in the summation for  $t_{(ijk)} = \frac{1}{3}(t_{123} + t_{231} + t_{132})$ , whereas those in  $t_{[ijk]} = 0$  cancel eachother.

**28.** Let  $\mathscr{B} = \{dt, dx, dy, dz\}$  be a coordinate basis of the 4-dimensional spacetime covector space  $V^*$  of  $V \sim \mathbb{R}^4$ . What is dim  $\bigwedge_p(V)$  for p = 0, 1, 2, 3, 4? Provide explicit bases  $\mathscr{B}_p$  of  $\bigwedge_p(V)$  for p = 0, 1, 2, 3, 4 induced by  $\mathscr{B}$ . We have, respectively, dim  $\mathscr{B}_0 = 1$ , dim  $\mathscr{B}_1 = 4$ , dim  $\mathscr{B}_2 = 6$ , dim  $\mathscr{B}_3 = 4$ , dim  $\mathscr{B}_4 = 1$ , with bases  $\mathscr{B}_0 = \{1\}$ ,  $\mathscr{B}_1 = \{dt, dx, dy, dz\} = \mathscr{B}$ ,  $\mathscr{B}_2 = \{dt \land dx, dt \land dy, dt \land dz, dx \land dy, dx \land dz, dy \land dz\}$ ,  $\mathscr{B}_3 = \{dt \land dx \land dy, dt \land dx \land dz, dt \land dy \land dz, dx \land dy \land dz\}$ ,  $\mathscr{B}_4 = \{dt \land dx \land dy \land dz\}$ .

**29.** Using the definition  $(\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k | \mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_k) = \det \langle \sharp \mathbf{v}_i, \mathbf{x}_j \rangle$ , show that for any  $\mathbf{v}, \mathbf{w} \in V$  (with  $\dim V \ge 2$ ),  $(\mathbf{v} \wedge \mathbf{w} | \mathbf{v} \wedge \mathbf{w}) \ge 0$ , with equality iff  $\mathbf{v} \wedge \mathbf{w} = 0$ . What if  $\dim V = 1$ ?

Hint: Recall the Schwartz inequality for inner products:  $|(\mathbf{v}|\mathbf{w})| \leq \sqrt{(\mathbf{v}|\mathbf{v})(\mathbf{w}|\mathbf{w})}$ .

We have

$$(\mathbf{v} \wedge \mathbf{w} | \mathbf{v} \wedge \mathbf{w}) = \det \begin{pmatrix} \langle \sharp \mathbf{v}, \mathbf{v} \rangle & \langle \sharp \mathbf{v}, \mathbf{w} \rangle \\ \langle \sharp \mathbf{w}, \mathbf{v} \rangle & \langle \sharp \mathbf{w}, \mathbf{w} \rangle \end{pmatrix} = \det \begin{pmatrix} (\mathbf{v} | \mathbf{v}) & (\mathbf{v} | \mathbf{w}) \\ (\mathbf{w} | \mathbf{v}) & (\mathbf{w} | \mathbf{w}) \end{pmatrix} = (\mathbf{v} | \mathbf{v}) (\mathbf{w} | \mathbf{w}) - (\mathbf{w} | \mathbf{v}) (\mathbf{v} | \mathbf{w}) \ge 0.$$

The last step follows from the Schwartz inequality, which also yields that equality occurs iff  $\mathbf{v} \propto \mathbf{w}$ , which is equivalent to  $\mathbf{v} \wedge \mathbf{w} = 0$ . Note that if dim V = 1 the inequality becomes a trivial equality, since then  $\mathbf{v} \wedge \mathbf{w} = 0$  for  $\mathbf{v}, \mathbf{w} \in V$ .

**30.** Cf. previous problem. Show that  $(\mathbf{v} \wedge \mathbf{w} | \mathbf{x} \wedge \mathbf{y}) = (\mathbf{x} \wedge \mathbf{y} | \mathbf{v} \wedge \mathbf{w})$  for all  $\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y} \in V$ .

We have

 $(\mathbf{v} \wedge \mathbf{w} | \mathbf{x} \wedge \mathbf{y}) = \det \begin{pmatrix} \langle \sharp \mathbf{v}, \mathbf{x} \rangle & \langle \sharp \mathbf{v}, \mathbf{y} \rangle \\ \langle \sharp \mathbf{w}, \mathbf{x} \rangle & \langle \sharp \mathbf{w}, \mathbf{y} \rangle \end{pmatrix} = \det \begin{pmatrix} (\mathbf{v} | \mathbf{x}) & (\mathbf{v} | \mathbf{y}) \\ (\mathbf{w} | \mathbf{x}) & (\mathbf{w} | \mathbf{y}) \end{pmatrix} \stackrel{*}{=} \det \begin{pmatrix} (\mathbf{x} | \mathbf{v}) & (\mathbf{x} | \mathbf{w}) \\ (\mathbf{y} | \mathbf{v}) & (\mathbf{y} | \mathbf{w}) \end{pmatrix} = \det \begin{pmatrix} \langle \sharp \mathbf{x}, \mathbf{v} \rangle & \langle \sharp \mathbf{x}, \mathbf{w} \rangle \\ \langle \sharp \mathbf{y}, \mathbf{v} \rangle & \langle \sharp \mathbf{y}, \mathbf{w} \rangle \end{pmatrix} = (\mathbf{x} \wedge \mathbf{y} | \mathbf{v} \wedge \mathbf{w}) .$ 

In step \* we have used symmetry of the inner product as well as the fact that det  $A = \det A^T$  for any square matrix A.

**31.** Expand and simplify the symmetrized holor  $g_{(ij}h_{k\ell})$  in terms of the unsymmetrized holor, i.e. in terms of terms like  $g_{ij}h_{k\ell}$  and similar ones with permuted index orderings, using the symmetry properties  $g_{ij} = g_{ji}$  and  $h_{ij} = h_{ji}$ .

There are 4! = 24 terms in total, which can be reduced to 4 identical copies of either of 6 terms as follows:  $g_{(ij}h_{k\ell}) = \frac{1}{4!} \sum_{\pi} g_{\pi(i)\pi(j)}h_{\pi(k)\pi(\ell)} = \frac{1}{6} (g_{ij}h_{k\ell} + g_{ik}h_{j\ell} + g_{i\ell}h_{jk} + g_{j\ell}h_{ik} + g_{jk}h_{i\ell}).$ 

**32.** Show that for the holor of any covariant 2-tensor we have  $T_{ij} = T_{(ij)} + T_{[ij]}$ .

**33.** Cf. the previous problem. Show that, for the holor of a typical covariant 3-tensor,  $T_{ijk} \neq T_{(ijk)} + T_{[ijk]}$ .

34. Expand and simplify the symmetrized holor  $g_{(ij}g_{k\ell)}$  in terms of the unsymmetrized holor, i.e. in terms of terms like  $g_{ij}g_{k\ell}$  and similar ones with permuted index orderings, using the symmetry property  $g_{ij} = g_{ji}$ .

There are 4! = 24 terms in total, which can be reduced to 8 identical copies of either of 3 terms as follows:  $g_{(ij}g_{k\ell)} = \frac{1}{4!}\sum_{\pi}g_{\pi(i)\pi(j)}g_{\pi(k)\pi(\ell)} = \frac{1}{3}(g_{ij}g_{k\ell} + g_{ik}g_{j\ell} + g_{i\ell}g_{jk})$ . This is a special case of the previous problem, with first and fourth, second and fifth, and third and sixth terms now being identical due to the identification  $g_{ij} = h_{ij}$ .

**35.** Consider the 1-form  $df = \partial_i f dx^i$  and define the gradient vector  $\nabla f = b df = \partial^i f \partial_i$  relative to the corresponding dual bases  $\{dx^i\}$  and  $\{\partial_i\}$ . Here  $\partial_i f$  is shorthand for the partial derivative  $\frac{\partial f}{\partial x^i}$ , whereas  $\partial^i f$  is a symbolic notation for the contravariant coefficient of  $\nabla f$  relative to  $\{\partial_i\}$  (all evaluated at a fiducial point). Show that  $\partial^i f = g^{ij}\partial_j f$ .

**36.** Cf. previous problem.

- Compute the matrix entries g<sup>ij</sup> explicitly for polar coordinates in n = 2, i.e. x̄<sup>1</sup> ≡ r and x̄<sup>2</sup> ≡ φ, starting from a standard inner product in Cartesian coordinates x<sup>1</sup> ≡ x = r cos φ, x<sup>2</sup> = y = r sin φ. Relative to basis {∂<sub>i</sub>} = {∂<sub>x</sub>, ∂<sub>y</sub>}, in which v = v<sup>i</sup>∂<sub>i</sub> = v<sup>x</sup>∂<sub>x</sub> + v<sup>y</sup>∂<sub>y</sub>, respectively w = w<sup>i</sup>∂<sub>i</sub> = w<sup>x</sup>∂<sub>x</sub> + w<sup>y</sup>∂<sub>y</sub>, the standard inner product takes the form (v|w) = η<sub>ij</sub>v<sup>i</sup>w<sup>j</sup> with η<sub>ij</sub> = 1 if i = j and 0 otherwise.
- Using your result, compute the components  $\overline{\partial}^1 f = \partial^r f$  and  $\overline{\partial}^2 f = \partial^{\phi} f$  of the gradient vector  $\nabla f = b df = \overline{\partial}^i f \overline{\partial}_i$  relative to the polar basis  $\{\overline{\partial}_i\} \cong \{\partial_r, \partial_{\phi}\}$ .

37. Show that the Kronecker symbol  $\delta_j^i$  is an invariant holor under the "tensor transformation law", and explain the attribute "invariant" in this context.

**38.** Cf. previous problem. Define  $\epsilon = \sqrt{g} \mu \in \bigwedge_n(V)$ , in which V is an inner product space,  $\mu = dx^1 \land \ldots \land dx^n$ , and  $g = \det g_{ij}$ , the determinant of the covariant metric tensor. Consider its expansion relative to two bases,  $\{dx^i\}$  respectively  $\{d\overline{x}^i\}$ , viz.  $\epsilon = \epsilon_{i_1...i_n} dx^{i_1} \otimes \ldots \otimes dx^{i_n} = \overline{\epsilon}_{i_1...i_n} d\overline{x}^{i_1} \otimes \ldots \otimes d\overline{x}^{i_n}$ . Find the relation between the respective holors  $\epsilon_{i_1...i_n}$  and  $\overline{\epsilon}_{i_1...i_n}$ . Is the holor of the  $\epsilon$ -tensor invariant in the same sense as above?

## 3. Differential Geometry

**1.** Show that the definitions (i)  $\nabla_{\partial_i}\partial_i = \Gamma_{ij}^k\partial_k$  and (ii)  $\Gamma_{ij}^k = \langle dx^k, \nabla_{\partial_i}\partial_i \rangle$  are equivalent.

Using (i) we have  $\langle dx^k, \nabla_{\partial_j} \partial_i \rangle = \langle dx^k, \Gamma^{\ell}_{ij} \partial_\ell \rangle = \Gamma^{\ell}_{ij} \langle dx^k, \partial_\ell \rangle = \Gamma^{\ell}_{ij} \delta^k_\ell = \Gamma^k_{ij}$ . Starting with (ii), since  $\nabla_{\partial_j} \partial_i \in \text{TM}$  there exist coefficients  $A^k_{ij}$  such that  $\nabla_{\partial_j} \partial_i = A^k_{ij} \partial_k$ . Insertion into the second slot of the Kronecker tensor yields  $\Gamma^k_{ij} = \langle dx^k, \nabla_{\partial_j} \partial_i \rangle = \langle dx^k, A^{\ell}_{ij} \partial_\ell \rangle = A^{\ell}_{ij} \langle dx^k, \partial_\ell \rangle = A^{\ell}_{ij} \delta^k_\ell = A^k_{ij} \delta^k_\ell = A^k_{ij}$ , which proves (i).

2. Show that if  $\Gamma_{ij}^k = \Gamma_{ji}^k$  in x-coordinates, then  $\overline{\Gamma}_{ij}^k = \overline{\Gamma}_{ji}^k$  in any coordinate basis induced by an arbitrary coordinate transformation  $x = x(\overline{x})$ .

This follows from the the fact that  $T_{ij}^k = \Gamma_{ji}^k - \Gamma_{ij}^k$  is the holor of a tensor, cf. the next problem.

**3.** Show that the holor  $T_{ij}^k = \Gamma_{ji}^k - \Gamma_{ij}^k$  "transforms as a  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ -tensor".

4. Show that the holor  $t_{\ell} = \Gamma_{k\ell}^k$  does not "transform as a covector", and derive its proper transformation law.

If 
$$T_{\ell}^m = \frac{\partial x^m}{\partial y^{\ell}}$$
 and  $S_r^k = \frac{\partial y^k}{\partial x^r}$ , then  $\bar{t}_{\ell} = T_{\ell}^m t_m + \frac{\partial^2 x^r}{\partial y^{\ell} \partial y^k} S_r^k$ . (The latter term can be rewritten as  $\frac{\partial^2 x^r}{\partial y^{\ell} \partial y^k} S_r^k = \frac{\partial}{\partial y^{\ell}} (\ln \det \frac{\partial x}{\partial y})$ .)

**5.** Consider 3-dimensional Euclidean space E furnished with Cartesian coordinates  $(x, y, z) \in \mathbb{R}^3$ . Assuming all Christoffel symbols  $\Gamma_{ij}^k$  vanish relative to the induced Cartesian basis  $(\partial_x, \partial_y, \partial_z)$ , derive the corresponding symbols  $\overline{\Gamma}_{ij}^k$  relative to a cylindrical coordinate system

$$\begin{cases} x = \rho \cos \phi \\ y = \rho \sin \phi \\ z = \zeta \end{cases}$$

Writing  $x^1 = x, x^2 = y, x^3 = z, y^1 = \rho, y^2 = \phi, y^3 = \zeta$ , and  $\partial_i = \frac{\partial}{\partial x^i}, \overline{\partial}_i = \frac{\partial}{\partial y^i}$ , the solution follows from  $\overline{\Gamma}_{ij}^k = S_p^k \overline{\partial}_j T_i^p$ , in which  $S_p^k = \frac{\partial y^k}{\partial x^p}$  and  $T_i^p = \frac{\partial x^p}{\partial y^i}$ . All symbols  $\overline{\Gamma}_{ij}^k$  vanish except  $\overline{\Gamma}_{\phi\phi}^\phi = \overline{\Gamma}_{\phi\phi}^\phi = \frac{1}{\rho}$  and  $\overline{\Gamma}_{\phi\phi}^\phi = -\rho$ .

6. Cf. the previous problem. Compute the components  $g_{ij}$  of the metric tensor for Euclidean 3-space E in cylindrical coordinates  $y^i$ , assuming a Euclidean metric  $g_{ij}(y)dy^i \otimes dy^j = \eta_{ij}dx^i \otimes dx^j$  in which the  $x^i$  are Cartesian coordinates, and  $\eta_{ij} = 1$  if i = j and  $\eta_{ij} = 0$  otherwise.

7. Cf. previous problems. Show that the Christoffel symbols  $\overline{\Gamma}_{ij}^k$  derived above are in fact those corresponding to the Levi-Civita connection of Euclidean 3-space E (in cylindrical coordinates), and provide the geodesic equations for a geodesic curve  $(\rho, \phi, \zeta) = (\rho(t), \phi(t), \zeta(t))$  in cylindrical coordinates. Finally, show that the solutions to these equations are indeed straight lines.

Hint: Compare the y-geodesic equations to the trivial x-geodesic equations  $\ddot{x} = \ddot{y} = \ddot{z} = 0$ .

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