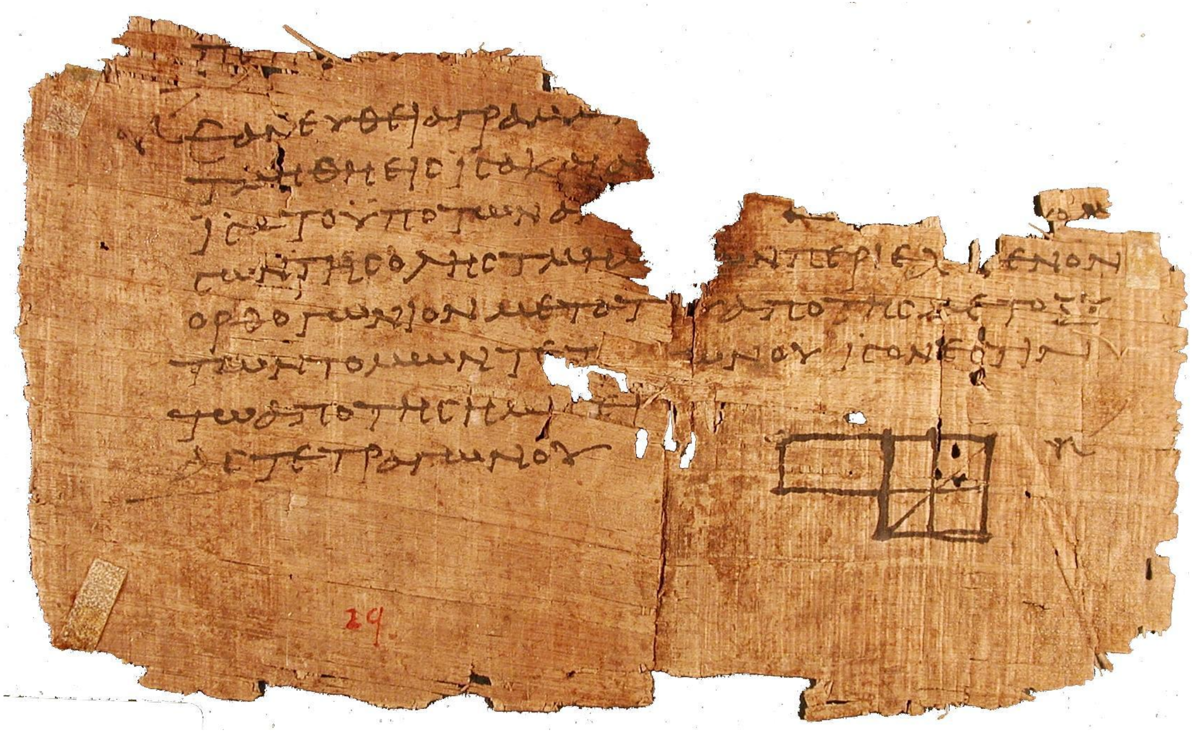
Problem Companion

Tensor Calculus and Differential Geometry

2WAH0



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March 25, 2015

Cover illustration: papyrus fragment from Euclid's Elements of Geometry, Book II [8].

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Preface

This problem companion belongs to the course notes “Tensor Calculus and Differential Geometry” (course code 2WAH0) by Luc Florack. Problems are concisely formulated. Einstein summation convention applies to all problems, unless stated otherwise. Please refer to the course notes for further details.

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Eindhoven, March 25, 2015.

1. Prerequisites from Linear Algebra

1. Let V be a real vector space, and $u \in V$. Show that $(-1) \cdot u = -u$, and $0 \cdot u = o$.

2. Show that $\mathscr{L}(V, W)$ is a vector space.

In analogy with the determinant of a matrix A, the so-called *permanent* is defined as

$$\operatorname{perm} A = \sum_{j_1,\ldots,j_n=1}^{n} |[j_1,\ldots,j_n]|\, A_{1j_1} \ldots A_{nj_n} = \frac{1}{n!} \sum_{\substack{i_1,\ldots,i_n=1 \\ j_1,\ldots,j_n=1}}^{n} |[i_1,\ldots,i_n]\,[j_1,\ldots,j_n]|\, A_{i_1 j_1} \ldots A_{i_n j_n}\,.$$

Note that the *nontrivial* factors among the weights $|[i_1,\ldots,i_n]|$ and $|[j_1,\ldots,j_n]|$ are invariably $+1$.

3. Can we omit the factors $|[i_1,\ldots,i_n]|$ and $|[j_1,\ldots,j_n]|$ in this definition of $\operatorname{perm} A$?

4. How many multiplications and additions/subtractions do you need in the numerical computation of $\det A$ and $\operatorname{perm} A$ for a generic $n \times n$ matrix A?

5. Given a generic $n \times n$ matrix A. Argue why its cofactor and adjugate matrices $\tilde{A}$, respectively $\tilde{A}^{\mathrm{T}}$, *always* exist, unlike its inverse A^{-1}. What is the condition for A^{-1} to be well-defined?

6. Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 1 \\ 6 & 3 & 8 \\ 8 & 7 & 10 \end{pmatrix}$$

Compute, *by hand*, $\det A$ and $\det B$. Likewise for $\operatorname{perm} A$ and $\operatorname{perm} B$.

7. Cf. previous problem. Compute the following expression involving the standard 3-dimensional inner and outer product of vectors (cf. the columns of B), and provide a geometrical interpretation of your result for $\det B$:

$$\left[\begin{pmatrix} 1 \\ 6 \\ 8 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ 8 \\ 10 \end{pmatrix}$$

8. Cf. previous problem. Compute the cofactor and adjugate matrices $\tilde{A}$, $\tilde{B}$, $\tilde{A}^{\mathrm{T}}$, and $\tilde{B}^{\mathrm{T}}$.

9. Cf. previous problem. Compute the inverse matrices A^{-1} and B^{-1}, if these exist.

10. Cf. previous problem. Compute $A\tilde{A}^{\mathrm{T}}$ and $B\tilde{B}^{\mathrm{T}}$.

11. Given $\det A$, $\det B$ for general $n \times n$ matrices A, B. What is $\det \tilde{A}$, $\det A^{\mathrm{T}}$, $\det(\lambda A)$, $\det(AB)$, and $\det A^k$ for $\lambda \in \mathbb{R}$, $k \in \mathbb{Z}$?

12. Consider the collection $A_{i_1\ldots i_n}$ for all $i_1,\ldots,i_n = 1,\ldots,n$. Suppose $A_{i_1\ldots i_k\ldots i_\ell\ldots i_n} = -A_{i_1\ldots i_\ell\ldots i_k\ldots i_n}$ for *any* $1 \leq k < \ell \leq n$ (complete antisymmetry). Show that $A_{i_1\ldots i_n} \propto [i_1 \ldots i_n]$.

13. Prove the following identities for the completely antisymmetric symbol in $n = 3$:

$$\begin{aligned}
[i,j,k]\,[\ell,m,n] &= \delta_{i\ell}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{k\ell} + \delta_{in}\delta_{j\ell}\delta_{km} - \delta_{im}\delta_{j\ell}\delta_{kn} - \delta_{i\ell}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{k\ell} \\
\sum_{i=1}^{3}[i,j,k]\,[i,m,n] &= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \\
\sum_{i,j=1}^{3}[i,j,k]\,[i,j,n] &= 2\delta_{kn} \\
\sum_{i,j,k=1}^{3}[i,j,k]\,[i,j,k] &= 6
\end{aligned}$$

$\star$ **14.** Compute the Gaussian integral $\gamma(A) = \int_{\mathbb{R}^n} \exp(-x^i A_{ij} x^j)\,dx$ for a symmetric positive definite $n \times n$ matrix A. Hint: There exists a rotation matrix R such that $R^{\mathrm{T}}AR = \Delta$, in which $\Delta = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$, with all $\lambda_p > 0$, $p = 1,\ldots,n$.

$\star$ **15.** Compute the extended Gaussian integral $\gamma(A,s) = \int_{\mathbb{R}^n} \exp(-x^i A_{ij} x^j + s_k x^k)\,dx$ for a symmetric positive definite $n \times n$ matrix A and arbitrary "source" $s \in \mathbb{R}^n$.

$\star$ **16.** Cf. previous problem. Consider the following integral: $\gamma^{i_1\cdots i_p}(A) = \int_{\mathbb{R}^n} x^{i_1} \ldots x^{i_p} \exp(-x^i A_{ij} x^j)\,dx$. Express $\gamma^{i_1\cdots i_p}(A)$ in terms of $\gamma(A,s)$. (You don't need to compute $\gamma^{i_1\cdots i_p}(A)$ explicitly.)

2. Tensor Calculus

17. Expand in $n = 2$ and $n = 3$ dimensions, respectively: $X_i Y^{ij} X_j$. Argue why we may assume, without loss of generality, that $Y^{ij} = Y^{ji}$ in this expression ("automatic" symmetry).

18. Expand in $n = 2$ and $n = 3$ dimensions, respectively: $X_{ij} Y^{ij}$. Are we allowed to assume that X_{ij} or Y^{ij} is symmetric in this expression?

19. Show that $\delta^i_i = n$ and $\delta^i_k \delta^k_j = \delta^i_j$.

20. Show that $\delta^i_j X^j_i = X^i_i$.

21. Suppose $x^i = A^i_j y^j$ and $y^i = B^i_j x^j$ for all $y \in \mathbb{R}^n$. Prove that $A^i_k B^k_j = \delta^i_j$.

22. Consider a Cartesian basis $\{\partial_1 \equiv \partial_x, \partial_2 \equiv \partial_y\}$ spanning a 2-dimensional plane. (Here ∂_i is shorthand for $\partial/\partial x^i$ if x^i denotes the i-th Cartesian coordinate; in $\mathbb{R}^2$ we identify $x^1 \equiv x$, $x^2 \equiv y$.) Let $\overline{x}^i$ denote the i-th polar coordinate, with $\overline{x}^1 \equiv r$ (radial distance) and $\overline{x}^2 \equiv \phi$ (polar angle), such that $x = r\cos\phi$, $y = r\sin\phi$. Assume $\overline{\partial}_j = A^i_j \partial_i$ (with $\overline{\partial}_1 \equiv \partial_r$, $\overline{\partial}_2 \equiv \partial_\phi$). Argue why this assumption holds, and compute the matrix A.

23. Cf. previous problem, but now for the relation between Cartesian and spherical coordinates in $\mathbb{R}^3$, defined by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, with radial distance $\overline{x}^1 \equiv r$, polar angle $\overline{x}^2 \equiv \theta$ and azimuthal angle $\overline{x}^3 \equiv \phi$, and, correspondingly, $\overline{\partial}_1 \equiv \partial_r$, $\overline{\partial}_2 \equiv \partial_\theta$, $\overline{\partial}_3 \equiv \partial_\phi$.

24. Cf. previous two problems. Consider the dual Cartesian bases $\{dx^1 \equiv dx, dx^2 \equiv dy\}$ in two dimensions, respectively $\{dx^1 \equiv dx, dx^2 \equiv dy, , dx^3 \equiv dz\}$ in three dimensions. Assume $dx^i = C^i_j d\overline{x}^j$, with dual bases $\{d\overline{x}^1 \equiv dr, d\overline{x}^2 \equiv d\phi\}$, respectively $\{d\overline{x}^1 \equiv dr, d\overline{x}^2 \equiv d\theta, , d\overline{x}^3 \equiv d\phi\}$. Show that $C = A$, i.e. the same matrix as computed in the previous two problems. (Notice the difference!)

25. Let $\mathbf{x} = x^i \mathbf{e}_i \in V$. Show that the map $\hat{\mathbf{a}} : V \to \mathbb{R}$ defined by $\hat{\mathbf{a}}(\mathbf{x}) = a_i x^i$ is a linear operator.

26. Cf. previous problem. A linear operator of this type is known as a *covector*, notation $\hat{\mathbf{a}} \in V^* \equiv \mathscr{L}(V, \mathbb{R})$. Explain why a covector $\hat{\mathbf{a}}$ (respectively covector space V^*) is formally a vector (respectively vector space).

27. Consider $\hat{\mathbf{a}} \in V^*$ with prototype $\hat{\mathbf{a}} : V \to \mathbb{R} : \mathbf{x} \mapsto \hat{\mathbf{a}}(\mathbf{x}) = a_i x^i$. Argue why this naturally provides an alternative interpretation of $\mathbf{x} \in V$ as an element of $V^{**} = (V^*)^* \sim V$.

28. Suppose $\langle \hat{\omega}, \mathbf{v} \rangle = C^i_j \omega_i v^j = \overline{C}^i_j \overline{\omega}_i \overline{v}^j$ relative to dual bases $\{\mathbf{e}_i, \hat{\mathbf{e}}^i\}$ (middle term), respectively $\{\mathbf{f}_i, \hat{\mathbf{f}}^i\}$ (right hand side), in which C^i_j and $\overline{C}^i_j$ are coefficients ("holors") to be determined. Show that the holor is basis independent, i.e. $C^i_j = \overline{C}^i_j$, and compute its components explicitly.

29. Suppose $\{\mathbf{e}_i\}$ and $\{\hat{\mathbf{e}}^i\}$ are dual bases of V, respectively V^*, and $\mathbf{e}_i = A^j_i \mathbf{f}_j$, $\hat{\mathbf{e}}^i = B^i_j \hat{\mathbf{f}}^j$ for some (a priori unrelated) transformation matrices A and B. Show that if $\{\mathbf{f}_j, \hat{\mathbf{f}}^j\}$ constitute dual bases then $B^j_k A^k_i = \delta^j_i$.

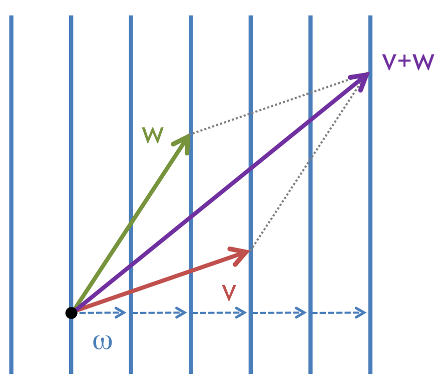
$\star$ **30.** Show that the following "length functional" $\mathscr{L}(\gamma)$ for a parameterized curve $\gamma : [T_-, T_+] \to \mathbb{R}^n : t \mapsto \gamma(t)$ with fixed end points $X_\pm = \gamma(T_\pm)$ is independent of the parametrization:

$$\mathscr{L}(\gamma) = \int_{T_-}^{T_+} \sqrt{g_{ij}(\gamma(t))\,\dot\gamma^i(t)\,\dot\gamma^j(t)}\,dt\,.$$

Here $\dot\gamma(t) = \dot\gamma^i(t)\mathbf{e}_i$ denotes the derivative of the curve, expanded relative to a fiducial basis $\{\mathbf{e}_i\}$, and $\gamma_{ij}(x)$ are the components of the inner product defined at base point $x \in \mathbb{R}^n$. To this end, consider a reparametrization of the form $s = s(t)$, with $\dot s(t) > 0$, say.

31. Cf. previous problem. The parameter s is called an *affine parameter* if, along the entire parameterized curve $\xi : [S_-, S_+] \to \mathbb{R}^n : s \mapsto \xi(s)$, we have $\|\dot\xi(s)\| = 1$ ("unit speed parameterization"), or $\left(\dot\xi(s)|\dot\xi(s)\right) = 1$, in which the l.h.s. pertains to the inner product at point $\xi(s) \in \mathbb{R}^n$. In other words, $g_{ij}(\xi(s))\,\dot\xi^i(s)\,\dot\xi^j(s) = 1$. Show that this can always be realized through suitable reparameterization starting from an *arbitrarily* parameterized curve $\gamma : [T_-, T_+] \to \mathbb{R}^n : t \mapsto \gamma(t)$.

32. The figure below shows a pictorial representation of vectors ($\mathbf{v}, \mathbf{w} \in V$) and covectors ($\boldsymbol{\omega} \in V^*$) in terms of graphical primitives. In this picture, a vector is denoted by a directed arrow, and a covector by an equally spaced set of level lines along a directed normal (i.e. "phase" increases in the direction of the directed normal, attaining consecutive integer values on the level lines drawn in the figure). Give a graphical interpretation of the contraction $\langle \boldsymbol{\omega}, \mathbf{v}\rangle \in \mathbb{R}$, and estimate from the figure the values of $\langle \boldsymbol{\omega}, \mathbf{v}\rangle$, $\langle \boldsymbol{\omega}, \mathbf{w}\rangle$, and $\langle \boldsymbol{\omega}, \mathbf{v}+\mathbf{w}\rangle$. Are these values consistent with the linearity of $\langle \boldsymbol{\omega}, \cdot\,\rangle$?



33. An inner product on V induces an inner product on V^*. Recall that for $\mathbf{x} = x^i\mathbf{e}_i \in V$ and $\mathbf{y} = y^j\mathbf{e}_j \in V$ we have $(\mathbf{x}|\mathbf{y}) = g_{ij}x^iy^j$. Let $\hat{\mathbf{x}} = x_i\hat{\mathbf{e}}^i \in V^*$, $\hat{\mathbf{y}} = y_j\hat{\mathbf{e}}^j \in V^*$, with $\langle \hat{\mathbf{e}}^i, \mathbf{e}_j\rangle = \delta^i_j$. Define $(\hat{\mathbf{x}}|\hat{\mathbf{y}})_* = (\flat\hat{\mathbf{x}}|\flat\hat{\mathbf{y}})$. Show that $(\hat{\mathbf{x}}|\hat{\mathbf{y}})_* = g^{ij}x_iy_j$, in which $g^{ik}g_{kj} = \delta^i_j$.

34. Let $\hat{\mathbf{u}}, \hat{\mathbf{v}} \in V^*$. Prove equivalence of nilpotency and antisymmetry of the wedge product, i.e. $\hat{\mathbf{u}} \wedge \hat{\mathbf{u}} = 0$ iff $\hat{\mathbf{u}} \wedge \hat{\mathbf{v}} = -\hat{\mathbf{v}} \wedge \hat{\mathbf{u}}$.

35. Let $\pi, \pi' : \{1, \ldots, p\} \to \{1, \ldots, p\} : k \mapsto \pi(k)$ be two bijections. By abuse of notation these can be identified with permutations on any symbolic set $\Omega_p = \{(a_1, \ldots, a_p)\}$ consisting of p-tuples of labeled symbols: $\pi : \Omega_p \to \Omega_p : (a_1, \ldots, a_p) \mapsto (a_{\pi(1)}, \ldots, a_{\pi(p)})$, and likewise for π'. Let $\pi(1, \ldots, p) = (\pi(1), \ldots, \pi(p))$, respectively $\pi'(1, \ldots, p) = (\pi'(1), \ldots, \pi'(p))$, argue that $\pi'' = \pi'\pi$, i.e. the (right-to-left) concatenation of π' and π, defines another permutation.

36. Argue that the concatenation $\pi'' = \pi'\pi$ of two permutations π' and π satisfies $\operatorname{sgn}\pi'' = \operatorname{sgn}\pi' \operatorname{sgn}\pi$.

37. Show that the symmetrisation map $\mathscr{S} : \mathbf{T}^0_p(V) \to \bigvee_p(V)$ is idempotent, i.e. $\mathscr{S} \circ \mathscr{S} = \mathscr{S}$.

38. Show that the antisymmetrisation map $\mathscr{A} : \mathbf{T}^0_p(V) \to \bigwedge_p(V)$ is idempotent, i.e. $\mathscr{A} \circ \mathscr{A} = \mathscr{A}$.

39. Cf. the two previous problems. Show that $\mathscr{A} \circ \mathscr{S} = \mathscr{S} \circ \mathscr{A} = 0 \in \mathscr{L}(\mathbf{T}^0_p(V), \mathbf{T}^0_p(V))$, i.e. the null operator on $\mathbf{T}^0_p(V)$ for $p \geq 2$. What if $p = 0, 1$?

40. Let $t_{i_1 \ldots i_p}$ be the holor of $\mathbf{T} \in \mathbf{T}^0_p(V)$. Show that the holor of $\mathscr{S}(\mathbf{T}) \in \bigwedge_p(V)$ is $t_{(i_1 \ldots i_p)} \stackrel{\text{def}}{=} \frac{1}{p!} \sum_\pi t_{\pi(i_1) \ldots \pi(i_p)}$.

41. Cf. previous problem. Show that the holor of $\mathscr{A}(\mathbf{T}) \in \bigwedge_p(V)$ is $t_{[i_1 \ldots i_p]} \stackrel{\text{def}}{=} \frac{1}{p!} \sum_\pi \operatorname{sgn} \pi \, t_{\pi(i_1) \ldots \pi(i_p)}$.

42. Write out the symmetrised holor $t_{(ijk)}$ in terms of t_{ijk}. Likewise for the antisymmetrised holor $t_{[ijk]}$.

43. Simplify the expressions of the previous problem as far as possible if t_{ijk} is known to possess the partial symmetry property $t_{ijk} = t_{jik}$.

44. Let $\mathscr{B} = \{dt, dx, dy, dz\}$ be a coordinate basis of the 4-dimensional spacetime covector space V^* of $V \sim \mathbb{R}^4$. What is $\dim \bigwedge_p(V)$ for $p = 0, 1, 2, 3, 4$? Provide explicit bases $\mathscr{B}_p$ of $\bigwedge_p(V)$ for $p = 0, 1, 2, 3, 4$ induced by $\mathscr{B}$.

45. Using the definition $(\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k | \mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_k) = \det\langle \sharp\mathbf{v}_i, \mathbf{x}_j \rangle$, show that for any $\mathbf{v}, \mathbf{w} \in V$ (with $\dim V \geq 2$), $(\mathbf{v} \wedge \mathbf{w} | \mathbf{v} \wedge \mathbf{w}) \geq 0$, with equality iff $\mathbf{v} \wedge \mathbf{w} = 0$. What if $\dim V = 1$?

Hint: Recall the Schwartz inequality for inner products: $|\,(\mathbf{v}|\mathbf{w})\,| \leq \sqrt{(\mathbf{v}|\mathbf{v})\,(\mathbf{w}|\mathbf{w})}$.

46. Cf. previous problem. Show that $(\mathbf{v} \wedge \mathbf{w} | \mathbf{x} \wedge \mathbf{y}) = (\mathbf{x} \wedge \mathbf{y} | \mathbf{v} \wedge \mathbf{w})$ for all $\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y} \in V$.

47. Expand and simplify the symmetrized holor $g_{(ij}h_{k\ell)}$ in terms of the unsymmetrized holor, i.e. in terms of terms like $g_{ij}h_{k\ell}$ and similar ones with permuted index orderings, using the symmetry properties $g_{ij} = g_{ji}$ and $h_{ij} = h_{ji}$.

48. Show that for the holor of any covariant 2-tensor we have $T_{ij} = T_{(ij)} + T_{[ij]}$.

49. Cf. the previous problem. Show that, for the holor of a typical covariant 3-tensor, $T_{ijk} \neq T_{(ijk)} + T_{[ijk]}$.

50. Expand and simplify the symmetrized holor $g_{(ij}g_{k\ell)}$ in terms of the unsymmetrized holor, i.e. in terms of terms like $g_{ij}g_{k\ell}$ and similar ones with permuted index orderings, using the symmetry property $g_{ij} = g_{ji}$.

51. Consider the 1-form $df = \partial_i f \, dx^i$ and define the gradient vector $\nabla f = \flat df = \partial^i f \, \partial_i$ relative to the corresponding dual bases $\{dx^i\}$ and $\{\partial_i\}$. Here $\partial_i f$ is shorthand for the partial derivative $\frac{\partial f}{\partial x^i}$, whereas $\partial^i f$ is a symbolic notation for the contravariant coefficient of ∇f relative to $\{\partial_i\}$ (all evaluated at a fiducial point). Show that $\partial^i f = g^{ij} \partial_j f$.

52. Cf. previous problem.

- Compute the matrix entries g^{ij} explicitly for polar coordinates in $n = 2$, i.e. $\overline{x}^1 \equiv r$ and $\overline{x}^2 \equiv \phi$, starting from a standard inner product in Cartesian coordinates $x^1 \equiv x = r \cos\phi$, $x^2 = y = r \sin\phi$. Relative to basis $\{\partial_i\} \widehat{=} \{\partial_x, \partial_y\}$, in which $\mathbf{v} = v^i \partial_i = v^x \partial_x + v^y \partial_y$, respectively $\mathbf{w} = w^i \partial_i = w^x \partial_x + w^y \partial_y$, the standard inner product takes the form $(\mathbf{v}|\mathbf{w}) = \eta_{ij} v^i w^j$ with $\eta_{ij} = 1$ if $i = j$ and 0 otherwise.

- Using your result, compute the components $\overline{\partial}^1 f = \partial^r f$ and $\overline{\partial}^2 f = \partial^\phi f$ of the gradient vector $\nabla f = \flat df = \overline{\partial}^i f \, \overline{\partial}_i$ relative to the polar basis $\{\overline{\partial}_i\} \widehat{=} \{\partial_r, \partial_\phi\}$.

53. Show that the Kronecker symbol δ^i_j is an invariant holor under the "tensor transformation law", and explain the attribute "invariant" in this context.

54. Cf. previous problem. Define $\boldsymbol{\epsilon} = \sqrt{g}\,\boldsymbol{\mu} \in \bigwedge_n(V)$, in which V is an inner product space, $\boldsymbol{\mu} = dx^1 \wedge \ldots \wedge dx^n$, and $g = \det g_{ij}$, the determinant of the covariant metric tensor. Consider its expansion relative to two bases, $\{dx^i\}$ respectively $\{d\overline{x}^i\}$, viz. $\boldsymbol{\epsilon} = \epsilon_{i_1 \ldots i_n}\, dx^{i_1} \otimes \ldots \otimes dx^{i_n} = \overline{\epsilon}_{i_1 \ldots i_n}\, d\overline{x}^{i_1} \otimes \ldots \otimes d\overline{x}^{i_n}$. Find the relation between the respective holors $\epsilon_{i_1 \ldots i_n}$ and $\overline{\epsilon}_{i_1 \ldots i_n}$. Is the holor of the $\boldsymbol{\epsilon}$-tensor invariant in the same sense as above?

3. Differential Geometry

55. Show that the definitions (i) $\nabla_{\partial_j}\partial_i = \Gamma^k_{ij}\partial_k$ and (ii) $\Gamma^k_{ij} = \langle dx^k, \nabla_{\partial_j}\partial_i\rangle$ are equivalent.

56. Show that if $\Gamma^k_{ij} = \Gamma^k_{ji}$ in x-coordinates, then $\overline{\Gamma}^k_{ij} = \overline{\Gamma}^k_{ji}$ in any coordinate basis induced by an arbitrary coordinate transformation $x = x(\overline{x})$.

57. Show that the holor $T^k_{ij} = \Gamma^k_{ji} - \Gamma^k_{ij}$ "transforms as a $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$-tensor".

58. Show that the holor $t_\ell = \Gamma^k_{k\ell}$ does not "transform as a covector", and derive its proper transformation law.

59. Consider 3-dimensional Euclidean space E furnished with Cartesian coordinates $(x, y, z) \in \mathbb{R}^3$. Assuming all Christoffel symbols Γ^k_{ij} vanish relative to the induced Cartesian basis $(\partial_x, \partial_y, \partial_z)$, derive the corresponding symbols $\overline{\Gamma}^k_{ij}$ relative to a cylindrical coordinate system

$$\left\{ \begin{array}{rcl} x & = & \rho\cos\phi \\ y & = & \rho\sin\phi \\ z & = & \zeta \end{array} \right.$$

60. Cf. the previous problem. Compute the components g_{ij} of the metric tensor for Euclidean 3-space E in cylindrical coordinates y^i, assuming a Euclidean metric $g_{ij}(y)dy^i \otimes dy^j = \eta_{ij}dx^i \otimes dx^j$ in which the x^i are Cartesian coordinates, and $\eta_{ij} = 1$ if $i = j$ and $\eta_{ij} = 0$ otherwise.

61. Cf. previous problems. Show that the Christoffel symbols $\overline{\Gamma}^k_{ij}$ derived above are in fact those corresponding to the Levi-Civita connection of Euclidean 3-space E (in cylindrical coordinates), and provide the geodesic equations for a geodesic curve $(\rho, \phi, \zeta) = (\rho(t), \phi(t), \zeta(t))$ in cylindrical coordinates. Finally, show that the solutions to these equations are indeed straight lines.

Hint: Compare the y-geodesic equations to the trivial x-geodesic equations $\ddot{x} = \ddot{y} = \ddot{z} = 0$.

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