EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0. Date: Tuesday July 2, 2019. Time: 18h00–21h00. Place: MATRIX 1.333.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin (in %).
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No other material or equipment may be used.
- The Einstein summation convention is in effect throughout this exam.

(30) 1. Γ-SYMBOL MISCELLANY.

We consider a Riemannian manifold M. The Gram matrix of the metric has components $g_{ij} = (\partial_i|\partial_j)$ relative to a coordinate basis. The connection $\nabla_\bullet$ of interest is the associated Levi-Civita connection. The Γ-symbols are defined by $\Gamma^k_{ij} = \langle dx^k, \nabla_{\partial_j}\partial_i\rangle$ relative to a coordinate basis. Square brackets surrounding indices in a holor denote antisymmetrization.

($7\frac{1}{2}$) **a.** Show that Γ^j_{ji} is not a basis independent covector holor.

($7\frac{1}{2}$) **b.** Show that $\Gamma^k_{[ij]}$ is a basis independent mixed tensor holor.

($7\frac{1}{2}$) **c.** Show that $\partial_i g = 2\Gamma^j_{ji} g$, in which g is the determinant of the Gram matrix.

($7\frac{1}{2}$) **d.** One defines the covariant derivative of g as $D_i g \doteq \partial_i g+$'$\Gamma$-correction terms'. Specify 'Γ-correction terms'.

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(15) 2. EXTERIOR DIFFERENTIALS.

Definition. The differential of an antisymmetric r-form field $\boldsymbol{\theta} = \theta_{|i_1\ldots i_r|}\, dx^{i_1} \wedge \ldots \wedge dx^{i_r}$ is given by

$$d\boldsymbol{\theta} = \partial_j \theta_{|i_1\ldots i_r|}\, dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_r}\,.$$

Definition. The commutator $[\mathbf{v}, \mathbf{w}]$ of the vector fields $\mathbf{v} = v^i\partial_i$ and $\mathbf{w} = w^i\partial_i$ satisfies

$$[\mathbf{v}, \mathbf{w}]f = v^j\partial_j(w^i\partial_i f) - w^j\partial_j(v^i\partial_i f)\,,$$

in which f is an arbitrary smooth function.

(5) **a.** Show that, for a smooth covector field $\boldsymbol{\zeta} = \zeta_i dx^i$, $d\boldsymbol{\zeta}(\mathbf{v}, \mathbf{w}) = \mathbf{v}(\boldsymbol{\zeta}(\mathbf{w})) - \mathbf{w}(\boldsymbol{\zeta}(\mathbf{v})) - \boldsymbol{\zeta}([\mathbf{v}, \mathbf{w}])$.

(5) **b.** Show that $d\boldsymbol{\zeta} = \partial_{[i}\zeta_{j]}\, dx^i \wedge dx^j$, in which the square brackets denote antisymmetrization.

(5) **c.** What is $d\boldsymbol{\zeta}$ for the case when $\boldsymbol{\zeta} = df$?

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(30) 3. Vector Bundle & Image Restoration.

Vector bundle: We generalize the notion of a tangent bundle to that of an arbitrary vector bundle

$$\mathrm{VM} = \cup_{x\in\mathrm{M}} \mathrm{VM}_x$$

in which each fiber VM_x at $x \in \mathrm{M}$ is a vector space of arbitrary dimension $\dim \mathrm{VM}_x = k \in \mathbb{N}$. By abuse of notation we write VM to either denote the vector bundle as such, or the set of smooth sections of VM, i.e. smooth vector fields of the form $v : \mathrm{M} \to \mathrm{VM} : x \mapsto v(x) \stackrel{\text{def}}{=} (x, v_x(x))$. The base manifold has dimension $\dim \mathrm{M} = n$. Note that $\mathrm{TM} = \cup_{x\in\mathrm{M}} \mathrm{TM}_x$ is a special case of a vector bundle, with $\dim \mathrm{TM}_x = n$.

Linear connection: A linear connection $\nabla : \mathrm{TM} \times \mathrm{VM} \to \mathrm{VM} : (v, \xi) \mapsto \nabla_v \xi$ on a vectorbundle VM is a map with the following properties, in which $f, g \in C^\infty(\mathrm{M})$, $v, w \in \mathrm{TM}$, $\xi, \eta \in \mathrm{VM}$:

(i.) $\nabla_{fv}\xi = f\, \nabla_v \xi$.

(ii.) $\nabla_{v+w}\xi = \nabla_v \xi + \nabla_w \xi$.

(iii.) $\nabla_v (f\,\xi + g\,\eta) = \nabla_v f\,\xi + f\,\nabla_v \xi + \nabla_v g\,\eta + g\,\nabla_v \eta$, in which $\nabla_v f \doteq df(v)$, $\nabla_v g \doteq dg(v)$.

Connection 1-forms: Let $\{e_a \in \mathrm{VM}\}_{a=1,\dots,k}$ be an ordered k-tuple of smooth sections such that $\{e_a|_x\}_{a=1,\dots,k}$ constitutes a basis of VM_x for each $x \in \mathrm{M}$, and

$$\nabla_v e_a \stackrel{\text{def}}{=} \omega_a^b(v) e_b \,.$$

Without vector argument, the symbols $\omega_a^b \in \mathrm{T}^*\mathrm{M}$ are referred to as the *connection 1-forms*.

(5) **a1.** Show that $\nabla_v \xi = (d\xi^a(v) + \omega_b^a(v)\xi^b) e_a$ (the 'covariant derivative of ξ along v').

(5) **a2.** Show that $\nabla_v \xi = v^i D_i \xi^a e_a \doteq v^i(\partial_i \xi^a + \omega_{bi}^a \xi^b) e_a$ for certain *connection coefficients* $\omega_{bi}^a \in \mathbb{R}$ $(a, b = 1, \dots, k,\ i = 1, \dots, n)$.

Covariant differential. We define the covariant differential $\nabla\xi \in \mathrm{T}^*\mathrm{M} \otimes \mathrm{VM}$ of $\xi \in \mathrm{VM}$ by

$$\nabla\xi \stackrel{\text{def}}{=} D\xi^a \otimes e_a \,, \quad \text{in which } D\xi^a \stackrel{\text{def}}{=} D_i \xi^a dx^i.$$

We henceforth consider the case of a 1-dimensional vector bundle, with single basis section $e \in \mathrm{VM}$, so that a general section takes the form $\Phi = \varphi\, e \in \mathrm{VM}$ for some smooth function $\varphi \in C^\infty(\mathrm{M})$.

We define the component $D\varphi$ of $\nabla\Phi$ relative to e by the identity $\nabla(\varphi\, e) \stackrel{\text{def}}{=} D\varphi \otimes e$.

(5) **b.** Show that $\nabla(\varphi\, e) = (d\varphi + \omega\varphi) \otimes e \doteq D\varphi \otimes e$, in which $\omega = \omega_i dx^i \in \mathrm{T}^*\mathrm{M}$ is the single connection 1-form, and $D_i\varphi = \partial_i\varphi + \omega_i\varphi$.

Consider a basis transformation on the 1-dimensional vector bundle, such that $\overline{e} = \lambda e$ is our new basis section, in which $\lambda \in C^\infty(\mathrm{M})$ is a positive scaling function.

(5) **c.** Requiring basis independence $\varphi\, e = \overline{\varphi}\, \overline{e}$, we stipulate $\nabla(\overline{\varphi}\, \overline{e}) = D\overline{\varphi} \otimes \overline{e}$, in which $D\overline{\varphi} = (d\overline{\varphi} + \overline{\omega}\, \overline{\varphi})$. Express the transformed connection 1-form $\overline{\omega} \in \mathrm{T}^*\mathrm{M}$ in terms of $\omega \in \mathrm{T}^*\mathrm{M}$ and $\lambda \in C^\infty(\mathrm{M})$.

We take $\mathrm{M} \sim \mathbb{R}^2$ to be the Euclidean plane, equipped with a standard inner product section on TM with globally constant Gram matrix $\eta_{ij} = 1$ if $i = j \in \{1, 2\}$, $\eta_{ij} = 0$ otherwise, in Cartesian coordinates $(x^1, x^2) = (x, y)$.

Laplacian on VM: We define the Laplacian of $\Phi = \varphi\, e \in \mathrm{VM}$ as $\mathbf{\Delta}\Phi \stackrel{\text{def}}{=} \eta^{ij}\nabla_{\partial_i}\nabla_{\partial_j}\Phi$. The induced covariant Laplacian of the holor φ is defined as $\Delta^{\text{cov}}\varphi \stackrel{\text{def}}{=} \eta^{ij} D_i D_j \varphi$, so that $\mathbf{\Delta}\Phi = (\Delta^{\text{cov}}\varphi)\, e$.

Image Restoration: Scratches in an image may be removed by *inpainting*. A standard approach is to delineate a region of interest $\Omega \subset \mathbb{R}^2$ containing the scratch, and to solve the Laplace equation $\Delta u = 0$ on its interior with Dirichlet boundary condition $u|_{\partial\Omega}$. The result is a graceful 'inpainting' of Ω. However, in textured regions, such as in the figure below, this method leaves visible scars, since it has no knowledge of any texture to be inpainted. To account for this, theoretical physicist Todor Georgiev from Adobe Photoshop stipulated the idea to exploit the 1-dimensional vector bundle construct above. The idea is to replace Δ by Δ^{cov} using a suitably defined connection $\omega = \omega_i dx^i$ (aka a 'gauge field' in physics), in other words, by interpreting an image as a section of VM rather than as a scalar field on M.

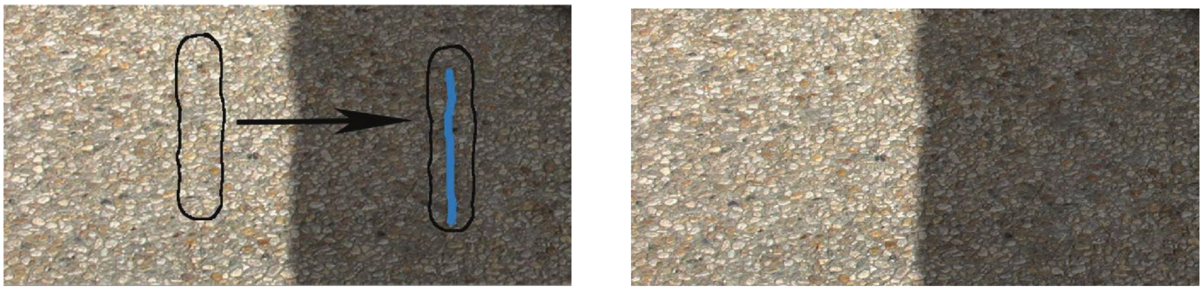
In order to remove a scratch, the user draws a contour $\partial\Omega$ around the region of interest Ω, then drags a copy $\partial\Omega'$ of that contour to delineate a region Ω' congruent to Ω elsewhere in the image (or in another image) containing a desirable texture for the inpainting.

(5) **d.** Associated with $\omega = \omega_i dx^i$ consider the vector-valued function $\vec{\omega} = (\omega_1, \omega_2) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Prove:

$$\Delta^{\text{cov}}\varphi = \Delta\varphi + (\operatorname{div}\vec{\omega} + \|\vec{\omega}\|^2)\varphi + 2\vec{\omega} \cdot \nabla\varphi\,,$$

in which $\Delta\varphi = \eta^{ij}\partial_i\partial_j\varphi$ denotes the standard Euclidean Laplacian of φ, $\operatorname{div}\omega = \eta^{ij}\partial_j\omega_i$ the standard divergence of $\vec{\omega}$, $\vec{\omega}\cdot\nabla\varphi = \eta^{ij}\omega_i\partial_j\varphi$ the standard directional derivative of φ along $\vec{\omega}$, and $\|\vec{\omega}\|^2 = \eta^{ij}\omega_i\omega_j$ the standard squared magnitude of $\vec{\omega}$.

(5) **e.** Determine the connection ω that renders the texture pattern φ inside Ω' 'covariantly constant', i.e. $D_i\varphi = 0$ on Ω'.



LEFT: IMAGE WITH SCRATCH INSIDE A MANUALLY DELINEATED REGION OF INTEREST, WITH A COPY OF ITS BOUNDARY SHIFTED TO A LOCATION CONTAINING THE DESIRED BACKGROUND TEXTURE FOR RESTORATION. RIGHT: RESULT AFTER 'COVARIANT INPAINTING'.

ADAPTED FROM: T. GEORGIEV, "COVARIANT DERIVATIVES AND VISION".
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(25) 4. PAULI EXCLUSION PRINCIPLE.

We consider two types of elementary particles, referred to as *bosons* and *fermions*. Such particles are characterized by an intrinsic property known as *spin*. The spin state of a particle is represented by a nontrivial vector in an abstract complex vector space Σ_s, in which $s \in \frac{1}{2}\mathbb{Z}_0^+ \stackrel{\text{def}}{=} \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ determines the dimension $\dim \Sigma_s = 2s + 1$. By definition, bosons have an integer-valued spin, so that $\dim \Sigma_s^{\text{boson}}$ is odd, whereas fermions have a half-integer valued spin, so that $\dim \Sigma_s^{\text{fermion}}$ is even.

There is a profound difference between bosons and fermions in the way systems of multiple identical particles behave, viz. (assuming physical states w.r.t. all other quantum numbers involved are identical):

Pauli's spin statistics theorem:

- A system of N identical bosons is represented by a vector space that is the *symmetric* N-fold tensor product $\bigvee^N(\Sigma_s) \stackrel{\text{def}}{=} \Sigma_s \otimes_S \ldots \otimes_S \Sigma_s$ of the 1-particle state space Σ_s.
- A system of N identical fermions is represented by a vector space that is the *antisymmetric* N-fold tensor product $\bigwedge^N(\Sigma_s) \stackrel{\text{def}}{=} \Sigma_s \otimes_A \ldots \otimes_A \Sigma_s$ of the 1-particle state space Σ_s.

Note: The infix product operators $\otimes_S$ and $\otimes_A$ are synonymous to $\vee$ and $\wedge$, respectively.

Let $\{e_\mu\}_{\mu=-s,-s+1,\ldots,s-1,s}$ denote a basis of Σ_s.

- For $s = \frac{1}{2}$ we simplify notation by writing $e_\downarrow$ and $e_\uparrow$ instead of $e_{-1/2}$, respectively $e_{+1/2}$.
- For $s = 1$ we simplify notation by writing e_-, e_0 and e_+ instead of e_{-1}, e_0, respectively e_{+1}.

We consider a system with $N \in \mathbb{N}$ identical fermions for the fermionic case with $s = \frac{1}{2}$.

(5) **a.** Provide an explicit basis of $\bigwedge^N(\Sigma_{1/2})$ for each $N \in \mathbb{N}$. Why cannot we put arbitrarily many identical fermions into a system? Support your argument by stating the explicit dimension of $\bigwedge^N(\Sigma_{1/2})$.

Next consider a system with $N \in \mathbb{N}$ identical bosons for the bosonic case with $s=1$.

(5) **b.** Can we put arbitrarily many identical bosons into a system? Support your argument by stating the explicit dimension of $\bigvee^N(\Sigma_1)$.

We consider the case of a system of N electrons ($s = \frac{1}{2}$ fermions) bound to an atom. The physical state of an electron is then determined by *four* quantum numbers, each of which labels one basis vector in a corresponding representation vector space:

- the *principal quantum number* $n \in \mathbb{N}$ (energy level, designating the principle electron shell),
- the *orbital angular momentum quantum number* $\ell \in \{0, \ldots, n-1\}$ (defining a subshell),
- the *magnetic quantum number* $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$ (defining an orbital), and
- the aforementioned *spin quantum number* $\mu \in \{-s, -s+1, \ldots, s-1, s\}$, in casu $\mu \in \{\downarrow, \uparrow\}$.

The complete representation space for a bound electron, V say, is just the product space of all four vector spaces. A basis vector of V is thus uniquely characterized by the four quantum numbers (n, ℓ, m, μ) subject to the indicated index range restrictions, and will be denoted by $|n, \ell, m, \mu\rangle$.

Note that $\dim V = \infty$ since there are infinitely many energy levels. We restrict ourselves to a fixed energy level $n \in \mathbb{N}$, and consider the corresponding subspace V_n.

Lemma. You may use the following sums without proof:

$$\sum_{\ell=0}^{n-1} \ell = \frac{1}{2} n(n-1)\,, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

(5) **c.** Determine $\dim V_n$.

According to Pauli's spin statistics theorem the representation vector space of a system of N bound electrons with fixed principal quantum number n is given by $V_n^N \stackrel{\text{def}}{=} \bigwedge^N(V_n) \stackrel{\text{def}}{=} V_n \otimes_A \ldots \otimes_A V_n$.

(5) **d1.** Determine the dimension $\dim V_n^N$ of V_n^N.

($2\frac{1}{2}$) **d2.** How many electrons maximally fit into the n^{th} principle shell of an atom?

($2\frac{1}{2}$) **d3.** How many electrons maximally fit into all principle shells up to (and including) the n^{th}?

Note: A fixed pair (n, ℓ) (with $0 \le \ell \le n-1$) determines the ℓ^{th} subshell of the n^{th} principle shell. Up to $\ell=3$, ℓ-values are known in the trade by the symbols s ($\ell=0$), p ($\ell=1$), d ($\ell=2$), respectively f ($\ell=3$).

THE END