# EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY 

Course code: 2WAH0-2WAH1. Date: Tuesday April 5, 2016. Time: 9h00-12h00. Place: AUD 11.

## Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- Einstein summation convention applies throughout for all repeated indices.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No equipment may be used.



## 1. Metric Connection

We consider an affine connection $\nabla: \mathrm{TM} \times \mathrm{TM} \rightarrow \mathrm{TM}$ on a Riemannian manifold M . In a local coordinate basis the connection is defined in terms of the Christoffel symbols:

$$
\nabla_{\partial_{j}} \partial_{i}=\Gamma_{i j}^{k} \partial_{k}
$$

We wish to complement this definition with (a) disambiguating rule(s) so as to single out a "metric connection", i.e. one uniquely determined by the Riemannian metric. The Riemannian metric is locally defined in terms of the components of the symmetric Gram matrix:

$$
g_{i j}=\left(\partial_{i} \mid \partial_{j}\right)
$$

As an ansatz we impose the following "metric compatibility condition":

$$
\nabla_{\mathbf{z}}(\mathbf{v} \mid \mathbf{w})=\left(\nabla_{\mathbf{z}} \mathbf{v} \mid \mathbf{w}\right)+\left(\mathbf{v} \mid \nabla_{\mathbf{z}} \mathbf{w}\right)
$$

for all vector fields $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathrm{TM}$
(15)
a. Show that

$$
g_{\ell j} T_{k i}^{\ell}+g_{i \ell} T_{k j}^{\ell}-g_{k \ell} S_{i j}^{\ell}=\partial_{k} g_{i j}-\partial_{j} g_{k i}-\partial_{i} g_{j k}
$$

in which $T_{i j}^{k}=\Gamma_{j i}^{k}-\Gamma_{i j}^{k}$ are the components of the torsion tensor, and $S_{i j}^{k}=\Gamma_{i j}^{k}+\Gamma_{j i}^{k}$. [HINT: CONSIDER z $=\partial_{k}, \mathbf{v}=\partial_{i}, \mathbf{w}=\partial_{j}$.]

To achieve complete disambiguation we impose the additional condition of torsion freeness:
(*) $\quad T_{i j}^{k}=0$.
(15) b. Show that these equations hold in any coordinate system.

$$
\text { [HINT: } \left.\Gamma_{i j}^{k}=\left\langle d x^{k} \mid \nabla_{\partial_{j}} \partial_{i}\right\rangle .\right]
$$

(10) c. Show that, with this condition ( $\star$ ) added, the Christoffel symbols $\Gamma_{i j}^{k}$ are uniquely determined, and provide their explicit form in terms of the metric coefficients $g_{i j}$ (and $g^{i j}$ ).

## 2. Infinitesimal Coordinate Transformations

We consider two coordinate systems on a common neighbourhood $\Omega$ of some fiducial point $\mathscr{P} \in \mathrm{M}$ on an $n$-dimensional Riemannian manifold M , with coordinates $x^{\mu}$ ("old") and $y^{\mu}$ ("new"), respectively. The coordinate transformation relating these systems is assumed to be "infinitesimal", in the sense that, for a fixed point $\mathscr{P} \in \mathrm{M}$,

$$
y^{\mu}(\mathscr{P})=x^{\mu}(\mathscr{P})+\epsilon \xi^{\mu}(\mathscr{P}),
$$

in which the $\xi^{\mu}$ are certain smooth functions on $\Omega$ and $0 \leq \epsilon \ll 1$. In all problems below we explicitly account for terms of $\mathcal{O}(\epsilon)$, but neglect effective terms of order $\mathcal{O}\left(\epsilon^{2}\right)$ (with $\approx$ indicating equalities up to such higher order terms).

On M we consider a scalar field $\phi$, a covector field $\omega=\omega_{\mu} d x^{\mu}$, and a cotensor field $T=T_{\mu \nu} d x^{\mu} \otimes d x^{\nu}$, in which $\left\{d x^{\mu}\right\}$ is dual to the coordinate basis $\left\{\partial_{\mu}\right\}$ of $\mathrm{TM}_{\mathscr{P}}$ in the $x^{\mu}$-system: $\left\langle d x^{\mu} \mid \partial_{\nu}\right\rangle=\delta_{\nu}^{\mu}$. With $\bar{\phi}$, $\bar{\omega}$, and $\bar{T}$ we denote the corresponding functional forms of these field quantities valid in the $y^{\mu}$-system. That is, by definition,

$$
\bar{\phi}(y(\mathscr{P}))=\phi(x(\mathscr{P})) \quad, \quad \bar{\omega}(y(\mathscr{P}))=\omega(x(\mathscr{P})) \quad, \text { resp. } \quad \bar{T}(y(\mathscr{P}))=T(x(\mathscr{P})) .
$$

All (component) functions are analytical, $\phi, \omega_{\mu}, T_{\mu \nu} \in C^{\omega}(\mathbf{M})$, admitting globally convergent Taylor expansions.
a. Show that $\bar{\phi} \approx \phi-\epsilon \xi^{\mu} \partial_{\mu} \phi$.
b. Show that $\bar{\omega}_{\mu} \approx \omega_{\mu}-\epsilon\left(\xi^{\nu} \partial_{\nu} \omega_{\mu}+\partial_{\mu} \xi^{\nu} \omega_{\nu}\right)$.
c. Show that $\bar{T}_{\mu \nu} \approx T_{\mu \nu}-\epsilon\left(\xi^{\rho} \partial_{\rho} T_{\mu \nu}+\partial_{\mu} \xi^{\rho} T_{\rho \nu}+\partial_{\nu} \xi^{\rho} T_{\mu \rho}\right)$.

If we assume that the manifold M is "almost Euclidean" in the sense of being only slightly curved, then we may stipulate a metric of the form $g=g_{\mu \nu} d x^{\mu} \otimes d x^{\nu}$, with

$$
g_{\mu \nu}=\eta_{\mu \nu}+\delta h_{\mu \nu}
$$

in which $\eta=\eta_{\mu \nu} d x^{\mu} \otimes d x^{\nu}$ is the flat Euclidean metric, $h=h_{\mu \nu} d x^{\mu} \otimes d x^{\nu}$ a smooth symmetric cotensor field of rank 2 , and $0 \leq \delta \ll 1$ a formal parameter. Moreover, we may assume that the components $\eta_{\mu \nu}$ are constant on $\Omega$ (i.e. all variability of $g_{\mu \nu}$ on $\Omega$ is accounted for by the $\mathcal{O}(\delta)$-term). In all problems below we explicitly account for terms of $\mathcal{O}(\delta)$ and $\mathcal{O}(\epsilon)$, but neglect effective terms of order $\mathcal{O}\left(\delta^{2}\right), \mathcal{O}(\delta \epsilon)$ and $\mathcal{O}\left(\epsilon^{2}\right)$.
d. Show that $\bar{g}_{\mu \nu} \approx g_{\mu \nu}-\epsilon\left(\partial_{\nu} \xi_{\mu}+\partial_{\mu} \xi_{\nu}\right)$.

Consider the "simultaneous contrast illusion" in the figure. The central bar has an overall constant physical value (all pixels inside the bar have the same numerical value). To appreciate this, cover its surroundings with blank paper (or with your hands), leaving only the bar visible, and observe that its perceptual brightness is indeed constant. We will refer to perceptual brightness under this conditioni.e. with context suppression-as physical brightness, as it agrees with the physical constancy of pixel values. Curiously, when regarded within its actual context (a background gradient), its perceptual brightness changes, revealing a smooth light-to-dark (left-to-right) transition along the bar.

We want to "explain" both phenomena-constant physical versus non-constant, context dependent perceptual brightness-in terms of one single geometric model.


THE BAR EXHIBITS A GRADIENT IN PERCEPTUAL BRIGHTNESS DESPITE CONSTANT PHYSICAL BRIGHTNESS.

Ignoring boundaries, the image domain is a 2-dimensional Euclidean space M, identified with $\mathbb{R}^{2}$ by employing a Cartesian coordinate system. Brightness is expressed as a dimensionfull scalar $\boldsymbol{u}=u \boldsymbol{\sigma}$, with dimensionless amplitude $u \geq 0$ relative to a dimensional unit $\boldsymbol{\sigma}$. We may interpret $\boldsymbol{\sigma}$ as a basis vector spanning a 1 -dimensional "brightness vector space" $\mathrm{U}=\operatorname{span}\{\boldsymbol{\sigma}\}$.

A physical image arises after associating a unique value to each point of M (a "pixel", with low/high values representing dark/bright regions, respectively). For this reason we model an image geometrically as a section of a fiber bundle. The fiber bundle itself comprises the set product $\mathrm{M} \times \mathrm{U}$. Given a fixed point $\boldsymbol{x} \in \mathrm{M}$ we refer to the set $\mathrm{U}_{\boldsymbol{x}}=\{(\boldsymbol{x}, \boldsymbol{u}) \mid \boldsymbol{u} \in \mathrm{U}\}$ as the fiber at $\boldsymbol{x} \in \mathrm{M}$. The fiber bundle is thus the union of fibers over all base points:

$$
\mathrm{M} \times \mathrm{U}=\cup_{x \in \mathrm{M}^{U}} \mathrm{U}_{\boldsymbol{x}}
$$

A section of this fiber bundle arises by picking a unique brightness value at every location in the image domain in a piecewise smooth fashion: $\{(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})=u(\boldsymbol{x}) \boldsymbol{\sigma}) \in \mathrm{M} \times \mathrm{U}\}$. This gives rise to a piecewise smooth scalar field $u: \mathrm{M} \rightarrow \mathbb{R}_{0}^{+}: \boldsymbol{x} \mapsto u(\boldsymbol{x})$, representing the numerical amplitude of the image $\boldsymbol{u}(\boldsymbol{x})=u(\boldsymbol{x}) \boldsymbol{\sigma}$ relative to the physical unit $\boldsymbol{\sigma}$. We will henceforth identify $\boldsymbol{x} \in \mathrm{M}$ with its coordinate pair $(x, y) \in \mathbb{R}^{2}$ in a given Cartesian coordinate system.

[^0]Spatial variations of physical and perceived brightness may now be accounted for by introducing a connection $\nabla$ on the fiber bundle, obeying the following product rule:

$$
\nabla \boldsymbol{u}=d u \boldsymbol{\sigma}+u \nabla \boldsymbol{\sigma} \quad \text { with } d u=\partial_{\mu} u d x^{\mu}=u_{x} d x+u_{y} d y .
$$

a2. Let $x^{\mu}=r_{\nu}^{\mu} \xi^{\nu}+a^{\mu}$ be an affine coordinate transformation, with new coordinates $\left(\xi^{1}, \xi^{2}\right)=(\xi, \eta)$. Which constraints must be imposed on $r_{\nu}^{\mu}$ and $a^{\mu}$ in order for these to be likewise Cartesian?

We stipulate a context dependent spatial variability of the perceptual brightness reference unit in terms of a connection $\nabla$ on $\mathrm{M} \times \mathrm{U}$ defined by some gauge field $A=A_{\mu} d x^{\mu} \in \mathrm{T}^{*} \mathrm{M}$ :

$$
\nabla \boldsymbol{\sigma} \stackrel{\text { def }}{=}\left(A_{\mu} d x^{\mu}\right) \boldsymbol{\sigma}=\left(A_{x} d x+A_{y} d y\right) \boldsymbol{\sigma} \quad \text { so that } \quad \nabla \boldsymbol{u} \stackrel{\text { def }}{=} D_{\mu} u d x^{\mu} \boldsymbol{\sigma}=\left(\partial_{\mu} u+A_{\mu} u\right) d x^{\mu} \boldsymbol{\sigma} .
$$

Global constancy of $\boldsymbol{\sigma}$ can be expressed as the "physical gauge" in which $\nabla \boldsymbol{\sigma}=0$, or $A_{\mu}=0$. In general, any gauge compatible with our (illusionary) visual percept may be called a "perceptual gauge".
b. Show that the gauge field $A=A_{\mu} d x^{\mu}$ is coordinate independent if the gauge field holor "transforms as a covector", i.e. show that after a general coordinate transformation, $x=x(\xi)$, its components $\bar{A}_{\mu}$ relative to the new $\xi$-coordinate system must satisfy

$$
\bar{A}_{\mu}=\frac{\partial x^{\rho}}{\partial \xi^{\mu}} A_{\rho}
$$

Recall the figure. We refer to the bar pattern as the foreground image (completed to cover the entire image plane by disregarding its context), $\boldsymbol{f}(\boldsymbol{x})=f(x, y) \boldsymbol{\sigma}$, and to the context pattern (completed to cover the entire image plane by disregarding the bar) as the background image, $\boldsymbol{b}(\boldsymbol{x})=b(x, y) \boldsymbol{\sigma}$, with brightness amplitudes $f(x, y)>0, b(x, y)>0$.

Since the background appears to generate the optical illusion in the foreground, we stipulate that the visual system fixes the gauge by "nullifying" context variations ("adaptation") in the sense that $\nabla \boldsymbol{b}=0$. For our stimulus we take $b(x, y)=e^{\beta(x)}$ independent of $y \in \mathbb{R}$, with $\beta(x) \in \mathbb{R}, \beta^{\prime}(x)>0, x \in \mathbb{R}$.
a1. For a scalar function $u$ the usual gradient $\nabla u=u^{\mu} \partial_{\mu}$ and differential $d u=u_{\mu} d x^{\mu}$ are related via the Euclidean metric by $\nabla u=b d u$. Explain why, and under which assumptions, we may identify their holors, i.e. $u_{\mu}=u^{\mu}$ (for any $u$ ).

We henceforth adopt coordinates $\left(x^{1}, x^{2}\right)=(x, y)$ such that co- and contravariant tensor components are identical (cf. a1). Such coordinates are referred to as Cartesian coordinates.

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c1. Compute the components of the gauge field $A_{\mu}(x, y)$ in terms of $\beta(x)$.
c2. Compute the components of the covariant derivative $D_{\mu} f(x, y)$ assuming $f(x, y)=c>0$, constant. How does your result "explain" the illusion?


[^0]:    ${ }^{1}$ Inspiration taken from: J. Koenderink. "The Brain a Geometry Engine". Psychol. Res. 1990, vol. 52, pp. 122-127, and T. Georgiev. "Covariant Derivatives and Vision". Proc. ECCV, 2006, Springer Verlag LNCS, vol. 3951, pp. 56-69.

