EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0-2WAH1. Date: Tuesday April 7, 2015. Time: 13h30–16h30. Place: AUD 9.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 2 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- Einstein summation convention applies throughout for all repeated indices.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No equipment may be used.

(**50**) **1.** GEODESICS

Consider the geodesic ordinary differential equations (o.d.e.'s) on a Riemannian space with coordinates coordinates $x \in \mathbb{R}^n$ for a parametrized curve $x = x(t)$ relative to the coordinate basis $\{\mathbf{e}_\mu = \partial_\mu\}_{\mu=1,\ldots,n}$:

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}(x)\dot{x}^\nu\dot{x}^\rho = 0\,. \qquad (*)$$

A (single/double) dot indicates a (1st/2nd order) derivative with respect to t. The Christoffel symbols $\Gamma^\mu_{\nu\rho}(x)$ represent the Levi-Civita connection compatible with the Riemannian metric with holor $g_{\mu\nu}(x)$.

(10) **a1.** Show that if $t = as + b$ for some new curve parameter $s \in \mathbb{R}$ and constants $a, b \in \mathbb{R}$, $a \neq 0$, then the geodesic equations assume the same form as those above $(*)$ when expressed in terms of s.

By the chain rule we have $\dfrac{d}{dt} = \dfrac{1}{a}\dfrac{d}{ds}$ and therefore $\dfrac{d^2}{dt^2} = \dfrac{1}{a^2}\dfrac{d^2}{ds^2}$. Substitution into the geodesic equation $(*)$, expressing the solution in terms of s, yields

$$\frac{1}{a^2}\left[\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{ds}\frac{dx^\rho}{ds}\right] = 0\,.$$

Ignoring the irrelevant factor $1/a^2$, this o.d.e. clearly has the same form as $(*)$.

(10) **a2.** Show that this is *not* true for a general reparametrization of the form $t = t(s)$, with $\dfrac{dt(s)}{ds} \neq 0$, by deriving the corresponding o.d.e.'s in terms of the general curve parameter s in this case.

By the same token the chain rule now gives $\frac{d}{dt} = \dot{s}\frac{d}{ds}$ and therefore $\frac{d^2}{dt^2} = \ddot{s}\frac{d}{ds} + \dot{s}^2\frac{d^2}{ds^2}$. Substitution into the geodesic equation $(*)$, expressing the solution in terms of s, yields

$$\dot{s}^2\left[\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{ds}\frac{dx^\rho}{ds}\right] + \ddot{s}\frac{dx^\mu}{ds} = 0\,. \qquad (\star)$$

This o.d.e. differs in form from $(*)$ due to the presence of the last term. Note that a1 is a special case.

A conformal metric transform is a pointwise scaling of the form $\overline{g}_{\mu\nu}(x) = e^{\alpha(x)}g_{\mu\nu}(x)$, in which $\alpha : \mathbb{R}^n \to \mathbb{R} : x \mapsto \alpha(x)$ denotes an arbitrary smooth scalar field.

(10) **b.** Show that the Christoffel symbols $\overline{\Gamma}^\mu_{\nu\rho}$ corresponding to the Levi-Civita connection induced by $\overline{g}_{\mu\nu}$ are given by

$$\overline{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + A^\mu_{\nu\rho} \quad \text{in which} \quad A^\mu_{\nu\rho} = \frac{1}{2}\left(\delta^\mu_\rho\partial_\nu\alpha + \delta^\mu_\nu\partial_\rho\alpha - g^{\mu\lambda}g_{\nu\rho}\partial_\lambda\alpha\right)\,.$$

By definition we have $\overline{\Gamma}^\mu_{\nu\rho} = \frac{1}{2}\overline{g}^{\mu\sigma}\left[\partial_\nu\overline{g}_{\sigma\rho} + \partial_\rho\overline{g}_{\nu\sigma} - \partial_\sigma\overline{g}_{\nu\rho}\right]$. Substituting $\overline{g}_{\mu\nu} = e^{\alpha(x)}g_{\mu\nu}$, and thus $\overline{g}^{\mu\nu} = e^{-\alpha(x)}g^{\mu\nu}$, yields the stated result after some elementary rewriting exploiting the product rule.

(5) **c1.** Provide the geodesic equations for $x^\mu = x^\mu(t)$ induced by $\{\overline{g}_{\mu\nu}, \overline{\Gamma}^\mu_{\nu\rho}\}$, expressed in $\{g_{\mu\nu}, \Gamma^\mu_{\nu\rho}, \alpha\}$.

The geodesic equation in this case is

$$\ddot{x}^\mu + \overline{\Gamma}^\mu_{\nu\rho}(x)\dot{x}^\nu\dot{x}^\rho = 0\,.$$

Substituting $\overline{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + A^\mu_{\nu\rho}$, recall b, yields

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}(x)\dot{x}^\nu\dot{x}^\rho + \dot{x}^\nu\partial_\nu\alpha\,\dot{x}^\mu - \frac{1}{2}\partial^\mu\alpha\, g_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0\,,$$

in which $\partial^\mu = g^{\mu\lambda}\partial_\lambda$. Note that this may also be written as

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}(x)\dot{x}^\nu\dot{x}^\rho + \dot{\alpha}\,\dot{x}^\mu - \frac{1}{2}\partial^\mu\alpha\,\|\dot{\mathbf{x}}\|^2 = 0\,. \qquad (**)$$

(5) **c2.** State the condition on the function α such that geodesics for the metric $\overline{g}_{\mu\nu}$ coincide with those for the unscaled metric $g_{\mu\nu}$ above $(*)$.
(*Caveat:* This does *not* mean that the geodesic equations take the same form as $(*)$, recall a2!)

The important observation is that we must compare $(**)$ to the geodesic o.d.e. for a *general* parameterization, i.e. to $(\star)$, which has the *same solutions* for the geodesic curves as $(*)$, and *not* to $(*)$, which holds only for affine parameterizations. Whereas it might be possible to remove the linear (second last) term on the l.h.s. of $**$ through a suitable reparameterization (viz. such that $\ddot{s}/\dot{s}^2 \propto \dot{\alpha}$), the quadratic (last) term cannot be done away with unless α happens to be a constant. In that case the geodesic equation reduces to $(*)$, and the solutions are clearly the same.

We now consider a differential manifold of Lorentzian type. This means that we drop the positivity axiom for the metric tensor, maintaining non-degeneracy of the inner product. More precisely:

Definition. A Lorentzian inner product on an n-dimensional linear space V is an indefinite symmetric bilinear mapping $(\,\cdot\,|\,\cdot\,) : V \times V \to \mathbb{R}$ which satisfies the following axioms:

- $\forall\, \mathbf{x}, \mathbf{y} \in V: \quad (\mathbf{x}|\mathbf{y}) = (\mathbf{y}|\mathbf{x})$,
- $\forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, \lambda, \mu \in \mathbb{R}: \quad (\lambda\mathbf{x} + \mu\mathbf{y}|\mathbf{z}) = \lambda\,(\mathbf{x}|\mathbf{z}) + \mu\,(\mathbf{y}|\mathbf{z})$,
- $\forall\, \mathbf{x} \in V$ with $\mathbf{x} \neq \mathbf{0}\ \exists\, \mathbf{y} \in V: \quad (\mathbf{x}|\mathbf{y}) \neq 0$.

Decomposing $\mathbf{x} = x^\mu \partial_\mu$, $\mathbf{y} = y^\mu \partial_\mu$, we have $(\mathbf{x}|\mathbf{y}) = G(\mathbf{x}, \mathbf{y}) = g_{\mu\nu} x^\mu y^\nu$, in which $g_{\mu\nu} = (\partial_\mu|\partial_\nu)$ are the components of the corresponding Gram matrix $\mathbf{G}$.

(5) **d.** Show that the Gram matrix $\mathbf{G}$ of a Lorentzian inner product is invertible.
(*Hint:* Hypothesize the existence of a null eigenvalue of $\mathbf{G}$.)

If $\mathbf{G}$ has a zero eigenvalue, then there exists a vector $\mathbf{x} \in V$, $\mathbf{x} \neq \mathbf{0}$ (viz. any corresponding eigenvector), such that $(\mathbf{x}|\mathbf{y}) = g_{\mu\nu} x^\mu y^\nu = 0$ for all $\mathbf{y} = y^\mu \partial_\mu \in V$. This contradicts the third (non-degeneracy) axiom for a Lorentzian inner product, thus $\mathbf{G}$ is invertible.

In the theory of general relativity a Lorentzian inner product in 4-dimensional spacetime, with signature $(- - - +)$, plays a prominent role, representing a tensor-valued gravitational potential. The signature pertains to the invariant number of positive $(+)$ and negative $(-)$ eigenvalues of the corresponding Gram matrix. Vectors satisfying $(\mathbf{x}|\mathbf{x}) = g_{\mu\nu} x^\mu x^\nu = 0$ are said to lie on the *light cone* and are referred to as *null vectors*. The light cone separates spacetime vectors into *spacelike* ($(\mathbf{x}|\mathbf{x}) < 0$) and *timelike* ($(\mathbf{x}|\mathbf{x}) > 0$) ones. The geodesic equations are formally identical to $(*)$, but with Levi-Civita Christoffel symbols pertaining to the Lorentzian rather than Riemannian metric.

(5) **e.** Explain the meaning of the general relativistic statement that "*null geodesics are invariant under conformal metric transforms*".
(*Hint:* Reconsider your answers under c1–c2 in the context of the stipulated Lorentzian metric.)

The meaning of this statement becomes apparent if you interpret the term "null geodesic" as one for which the *tangent vector* is (everywhere) a null vector, i.e. $(\dot{\mathbf{x}}|\dot{\mathbf{x}}) = \|\dot{\mathbf{x}}\|^2 = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$. (Due to the non-positive nature of the Lorentzian inner product this no longer implies $\dot{x}^\mu = 0$, as opposed to the genuine Riemannian case before.) If this is the case, then the last term in $(**)$ is absent, recall c1. As noticed before (recall c2), the term $\dot{\alpha}\, \dot{x}^\mu$ can be matched against the term $\dfrac{\ddot{s}}{\dot{s}^2} \dfrac{dx^\mu}{ds}$ in $(\star)$ through a suitable reparameterization, which shows that the solutions of $(**)$ are in fact invariant under conformal metric transforms. This argument clearly does not hold for spacelike or timelike geodesics.

♣

(50) 2. CARTOGRAPHY

In this problem we model the earth's surface as the unit sphere $\mathbb{S} \subset \mathbb{E}$ embedded in Euclidean space $\mathbb{E} = \mathbb{R}^3$ by parametrizing the Cartesian coordinates $(x^1, x^2, x^3) = (x, y, z) \in \mathbb{R}^3$ of a point $\mathscr{P} \in \mathbb{S}$ in terms of polar angles $(u^1, u^2) = (\phi, \theta)$, as follows:

$$E : \mathbb{S} \to \mathbb{E} : (\phi, \theta) \mapsto (x = E^1(\phi,\theta), y = E^2(\phi,\theta), z = E^3(\phi,\theta)) : \begin{cases} E^1(\phi,\theta) &= \cos\phi\cos\theta \\ E^2(\phi,\theta) &= \sin\phi\cos\theta \\ E^3(\phi,\theta) &= \sin\theta\ , \end{cases}$$

with $(\phi, \theta) \in (-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. This mapping naturally induces *push-forward* and *pull-back* maps, $E_* : \mathrm{T}\mathbb{S} \to \mathrm{T}\mathbb{E} : \mathbf{v} \mapsto E_*\mathbf{v}$, respectively $E^* : \mathrm{T}^*\mathbb{E} \to \mathrm{T}^*\mathbb{S} : \boldsymbol{\omega} \mapsto E^*\boldsymbol{\omega}$, in which $\mathbf{v} = v^a \partial_a \in \mathrm{T}\mathbb{S}$ and $\boldsymbol{\omega} = \omega_i dx^i \in \mathrm{T}^*\mathbb{E}$. These are defined via linear extension through their actions on basis (co-)vectors:

$$E_*\partial_a = E^i_a \partial_i \quad \text{respectively} \quad E^* dx^i = E^i_a du^a \quad \text{with Jacobian} \quad E^i_a = \frac{\partial E^i}{\partial u^a} \, .$$

Notation. Summation convention applies to repeated indices. We use shorthands $\partial_a = \frac{\partial}{\partial u^a}$ and $\partial_i = \frac{\partial}{\partial x^i}$. Indices $a, b, c, \ldots = 1, 2$ from the beginning of the alphabet pertain to the 2-dimensional unit sphere $\mathbb{S}$, indices $i, j, k, \ldots = 1, 2, 3$ from the middle of the alphabet to the 3-dimensional Euclidean space $\mathbb{E}$.

(5) **a1.** Express each of the 1-forms E^*dx, E^*dy, and E^*dz in terms of $d\phi$ and $d\theta$.

We have

$$\begin{aligned} E^*dx &= -\sin\phi\cos\theta\, d\phi - \cos\phi\sin\theta\, d\theta \\ E^*dy &= \cos\phi\cos\theta\, d\phi - \sin\phi\sin\theta\, d\theta \\ E^*dz &= \cos\theta\, d\theta \end{aligned}$$

(5) **a2.** Express each of the vectors $E_*\partial_\phi$, and $E_*\partial_\theta$ in terms of ∂_x, ∂_y, and ∂_z.

$$\begin{aligned} E_*\partial_\phi &= -\sin\phi\cos\theta\, \partial_x + \cos\phi\cos\theta\, \partial_y \\ E_*\partial_\theta &= -\cos\phi\sin\theta\, \partial_x - \sin\phi\sin\theta\, \partial_y + \cos\theta\, \partial_z \end{aligned}$$

The coefficients may be expressed in terms of x, y, z with the help of the following identities:

$$\cos\phi = \frac{x}{\sqrt{1-z^2}} \qquad \sin\phi = \frac{y}{\sqrt{1-z^2}} \qquad \cos\theta = \sqrt{1-z^2} \qquad \sin\theta = z\, .$$

This yields:

$$\begin{aligned} E_*\partial_\phi &= -y\, \partial_x + x\, \partial_y \\ E_*\partial_\theta &= -\frac{xz}{\sqrt{1-z^2}}\, \partial_x - \frac{yz}{\sqrt{1-z^2}}\, \partial_y + \sqrt{1-z^2}\, \partial_z \end{aligned}$$

The Euclidean metric tensor on $\mathbb{E}$ is given by $\mathbf{h} = h_{ij}\, dx^i \otimes dx^j = dx \otimes dx + dy \otimes dy + dz \otimes dz$. This metric naturally induces a *pull-back metric* $\mathbf{g} = E^*\mathbf{h} = g_{ab}\, du^a \otimes du^b = (h_{ij} \circ E)\, E^*dx^i \otimes E^*dx^j$ on $\mathbb{S}$. The symbol $\circ$ denotes composition, i.e. $(h_{ij} \circ E)(\phi, \theta) = h_{ij}(E(\phi, \theta))$.

(5) **b1.** Show that, in general, $g_{ab}\, du^a \otimes du^b = \dfrac{\partial E^i}{\partial u^a}(h_{ij} \circ E)\dfrac{\partial E^j}{\partial u^b}\, du^a \otimes du^b$.

Using $E^*\mathbf{h} = (h_{ij} \circ E)\, E^*dx^i \otimes E^*dx^j$ and $E^*dx^i = E^i_a du^a$ we get $E^*\mathbf{h}(\phi, \theta) = h_{ij}(E(\phi, \theta)) E^i_a(u) E^j_b(u) du^a \otimes du^b$, in which we recognize the above holor.

(5) **b2.** Show that, for the particular choice of the mapping E above, $\mathbf{g} = \cos^2\theta d\phi \otimes d\phi + d\theta \otimes d\theta$.

Substitution of the results found in a1 into the general expression of b1, recalling that $h_{11} = h_{22} = h_{33} = 1$ and $h_{ij} = 0$ if $i \neq j$, yields (with abbreviations s and c denoting sin, respectively cos)

$\mathbf{g} = (-\mathrm{s}\phi\mathrm{c}\theta\, d\phi - \mathrm{c}\phi\mathrm{s}\theta\, d\theta)\otimes(-\mathrm{s}\phi\mathrm{c}\theta\, d\phi - \mathrm{c}\phi\mathrm{s}\theta\, d\theta) + (\mathrm{c}\phi\mathrm{c}\theta\, d\phi - \mathrm{s}\phi\mathrm{s}\theta\, d\theta)\otimes(\mathrm{c}\phi\mathrm{c}\theta\, d\phi - \mathrm{s}\phi\mathrm{s}\theta\, d\theta) + (\mathrm{c}\theta\, d\theta)\otimes(\mathrm{c}\theta\, d\theta) = \mathrm{c}^2\theta d\phi\otimes d\phi + d\theta\otimes d\theta$.

The Christoffel symbols on $\mathbb{S}$ in u-coordinates are given by $\Gamma^c_{ab} = \dfrac{1}{2} g^{c\ell}\left(\partial_a g_{\ell b} + \partial_b g_{a\ell} - \partial_\ell g_{ab}\right)$.

($7\frac{1}{2}$) **c1.** Compute the following Christoffel symbols on $\mathbb{S}$: (i) $\Gamma^\phi_{\phi\theta}$, (ii) $\Gamma^\phi_{\theta\phi}$, and (iii) $\Gamma^\theta_{\phi\phi}$.

These are in fact the only non-vanishing Christoffel symbols: $\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = -\tan\theta$ and $\Gamma^\theta_{\phi\phi} = \sin\theta\cos\theta$.

($2\frac{1}{2}$) **c2.** Pick *one* of the *remaining* symbols Γ^c_{ab} not considered in problem c1, and show that it vanishes.

This follows by careful inspection. What you need to exploit in your proof are, besides the definition of the Γ-symbols, the special properties of the metric tensor, viz. the fact that (i) the matrix with coefficients g_{ab} is diagonal, i.e. $g_{\theta\phi} = g_{\phi\theta} = 0$, (ii) $g_{\phi\phi}$ does not depend on ϕ, and (iii) $g_{\theta\theta}$ is a global constant.

(5) **d1.** Provide the geodesic equations for a parametrized geodesic on $\mathbb{S}$: $(\phi, \theta) = (\phi(t), \theta(t))$, with $t \in \mathbb{R}$.

In general $\ddot{u}^c + \Gamma^c_{ab}\dot{u}^a\dot{u}^b = 0$, thus, using c1, $\ddot{\phi} + 2\Gamma^\phi_{\phi\theta}\dot{\phi}\dot{\theta} = 0$ and $\ddot{\theta} + \Gamma^\theta_{\phi\phi}\dot{\phi}^2 = 0$, i.e. $\ddot{\phi} - 2\tan\theta\dot{\phi}\dot{\theta} = 0$ and $\ddot{\theta} + \sin\theta\cos\theta\dot{\phi}^2 = 0$.

(5) **d2.** Show that the equator is a geodesic, and argue why this implies that *all* great circles (intersections of planes with $\mathbb{S}$ through the earth's center) are geodesics.

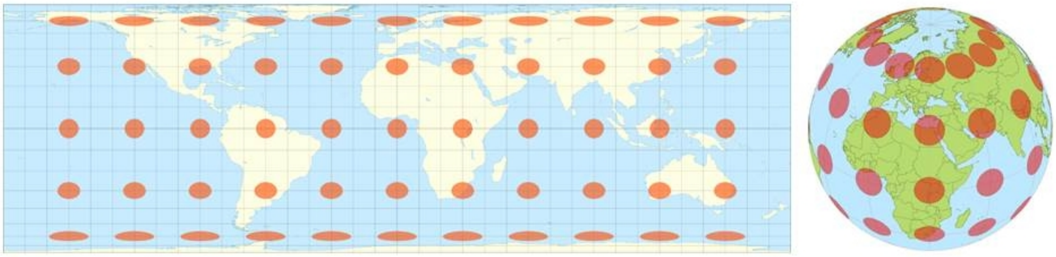
One trivial solution is $\theta(t) = 0$ and $\phi(t) = \phi_0 + \omega t$, with integration constants $\phi_0, \omega \in \mathbb{R}$. For $\omega \neq 0$ this is the equator, which is a great circle. By spherical symmetry this implies that *any* great circle must be a geodesic.

LAMBERT CYLINDRICAL EQUAL-AREA PROJECTION (JOHANN HEINRICH LAMBERT, 1728–1777).

The *Lambert cylindrical equal-area projection*

$$L : \mathbb{S} \to \mathbb{M} : (\phi, \theta) \mapsto (\xi = L^1(\phi, \theta), \eta = L^2(\phi, \theta)) : \left\{ \begin{array}{lcl} L^1(\phi, \theta) & = & \phi \\ L^2(\phi, \theta) & = & \sin\theta \end{array} \right. ,$$

with $(\phi, \theta) \in (-\pi, \pi] \times [-\dfrac{\pi}{2}, \dfrac{\pi}{2}]$, provides a flat chart $\mathbb{M}$ of the earth's surface $\mathbb{S}$, cf. the figure.



TISSOT'S INDICATRICES OF DEFORMATION ARE USED TO VISUALIZE SURFACE DISTORTIONS DUE TO PROJECTION.

The unit volume form on an n-dimensional Riemannian space is given by $\boldsymbol{\epsilon} = \sqrt{g}\, dz^1 \wedge \ldots \wedge dz^n$ in any coordinate basis $\{dz^1, \ldots, dz^n\}$, in which g denotes the determinant of the covariant metric tensor. Below we refer to a 2-dimensional volume form as an "area form".

(5) **e1.** Show that the unit area 2-form on the sphere $\mathbb{S}$ is given by $\boldsymbol{\epsilon}_{\mathbb{S}} = \cos\theta d\phi \wedge d\theta$.

From b2 it follows that $g = \cos^2\theta$, so $\sqrt{g} = \cos\theta$ (the sign follows from the domain of definition, with $-\pi/2 \leq \theta \leq \pi/2$), so with the identifications $u^1 = \phi$, $u^2 = \theta$ we find $\boldsymbol{\epsilon}_{\mathbb{S}} = \sqrt{g}\, du^1 \wedge du^2 = \cos\theta d\phi \wedge d\theta$.

We endow $\mathbb{M}$ with a 2-dimensional Euclidean structure with Cartesian coordinates $(\xi^1 = \xi, \xi^2 = \eta)$. In terms of these coordinates the metric on $\mathbb{M}$ takes the form $\boldsymbol{\eta} = \eta_{\mu\nu} d\xi^\mu \otimes d\xi^\nu = d\xi \otimes d\xi + d\eta \otimes d\eta$, with trivial metric determinant $\eta = 1$.

(5) **e2.** Prove that Lambert cylindrical equal-area projection preserves areas, as follows. Show that the area form induced by pulling back the *unit* area form on the earth's map onto the earth's surface under L^*, i.e. $\boldsymbol{\omega}_{\mathbb{S}} = L^* \boldsymbol{\epsilon}_{\mathbb{M}} \stackrel{\text{def}}{=} \sqrt{\eta \circ L}\, L^* d\xi \wedge L^* d\eta$, is in fact the *unit* area form on $\mathbb{S}$, i.e. show that $\boldsymbol{\omega}_{\mathbb{S}} = \boldsymbol{\epsilon}_{\mathbb{S}}$.

Using $\eta = 1$, $L^* d\xi^\mu = L^\mu_a du^a$, with $L^\mu_a = \frac{\partial L^\mu}{\partial u^a}$, we get $L^* d\xi = d\phi$ and $L^* d\eta = \cos\theta d\theta$, whence $\boldsymbol{\omega}_{\mathbb{S}} = \cos\theta d\phi \wedge d\theta = \boldsymbol{\epsilon}_{\mathbb{S}}$, cf. e1.

THE END