EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0-2WAH1. Date: Tuesday April 7, 2015. Time: 13h30–16h30. Place: AUD 9.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 2 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- Einstein summation convention applies throughout for all repeated indices.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No equipment may be used.

(**50**) **1.** GEODESICS

Consider the geodesic ordinary differential equations (o.d.e.'s) on a Riemannian space with coordinates coordinates $x \in \mathbb{R}^n$ for a parametrized curve $x = x(t)$ relative to the coordinate basis $\{\mathbf{e}_\mu = \partial_\mu\}_{\mu=1,\ldots,n}$:

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}(x)\dot{x}^\nu \dot{x}^\rho = 0\,. \qquad (*)$$

A (single/double) dot indicates a (1^{st}/2^{nd} order) derivative with respect to t. The Christoffel symbols $\Gamma^\mu_{\nu\rho}(x)$ represent the Levi-Civita connection compatible with the Riemannian metric with holor $g_{\mu\nu}(x)$.

(10) **a1.** Show that if $t = as + b$ for some new curve parameter $s \in \mathbb{R}$ and constants $a, b \in \mathbb{R}$, $a \neq 0$, then the geodesic equations assume the same form as those above $(*)$ when expressed in terms of s.

(10) **a2.** Show that this is *not* true for a general reparametrization of the form $t = t(s)$, with $\dfrac{dt(s)}{ds} \neq 0$, by deriving the corresponding o.d.e.'s in terms of the general curve parameter s in this case.

A conformal metric transform is a pointwise scaling of the form $\overline{g}_{\mu\nu}(x) = e^{\alpha(x)} g_{\mu\nu}(x)$, in which $\alpha : \mathbb{R}^n \to \mathbb{R} : x \mapsto \alpha(x)$ denotes an arbitrary smooth scalar field.

(10) **b.** Show that the Christoffel symbols $\overline{\Gamma}^\mu_{\nu\rho}$ corresponding to the Levi-Civita connection induced by $\overline{g}_{\mu\nu}$ are given by

$$\overline{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + A^\mu_{\nu\rho} \quad \text{in which} \quad A^\mu_{\nu\rho} = \frac{1}{2}\left(\delta^\mu_\rho \partial_\nu \alpha + \delta^\mu_\nu \partial_\rho \alpha - g^{\mu\lambda} g_{\nu\rho} \partial_\lambda \alpha\right)\,.$$

(5) **c1.** Provide the geodesic equations for $x^\mu = x^\mu(t)$ induced by $\{\overline{g}_{\mu\nu}, \overline{\Gamma}^\mu_{\nu\rho}\}$, expressed in $\{g_{\mu\nu}, \Gamma^\mu_{\nu\rho}, \alpha\}$.

(5) **c2.** State the condition on the function α such that geodesics for the metric $\overline{g}_{\mu\nu}$ coincide with those for the unscaled metric $g_{\mu\nu}$ above $(*)$.
(*Caveat:* This does *not* mean that the geodesic equations take the same form as $(*)$, recall a2!)

We now consider a differential manifold of Lorentzian type. This means that we drop the positivity axiom for the metric tensor, maintaining non-degeneracy of the inner product. More precisely:

Definition. A Lorentzian inner product on an n-dimensional linear space V is an indefinite symmetric bilinear mapping $(\,\cdot\,|\,\cdot\,) : V \times V \to \mathbb{R}$ which satisfies the following axioms:

- $\forall\, \mathbf{x}, \mathbf{y} \in V : \quad (\mathbf{x}|\mathbf{y}) = (\mathbf{y}|\mathbf{x})$,
- $\forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, \lambda, \mu \in \mathbb{R} : \quad (\lambda\mathbf{x} + \mu\mathbf{y}|\mathbf{z}) = \lambda\,(\mathbf{x}|\mathbf{z}) + \mu\,(\mathbf{y}|\mathbf{z})$,
- $\forall\, \mathbf{x} \in V$ with $\mathbf{x} \neq \mathbf{0}\ \exists\, \mathbf{y} \in V : \quad (\mathbf{x}|\mathbf{y}) \neq 0$.

Decomposing $\mathbf{x} = x^\mu \partial_\mu$, $\mathbf{y} = y^\mu \partial_\mu$, we have $(\mathbf{x}|\mathbf{y}) = G(\mathbf{x}, \mathbf{y}) = g_{\mu\nu} x^\mu y^\nu$, in which $g_{\mu\nu} = (\partial_\mu|\partial_\nu)$ are the components of the corresponding Gram matrix $\mathbf{G}$.

(5) **d.** Show that the Gram matrix $\mathbf{G}$ of a Lorentzian inner product is invertible.
(*Hint:* Hypothesize the existence of a null eigenvalue of $\mathbf{G}$.)

In the theory of general relativity a Lorentzian inner product in 4-dimensional spacetime, with signature $(- - - +)$, plays a prominent role, representing a tensor-valued gravitational potential. The signature pertains to the invariant number of positive $(+)$ and negative $(-)$ eigenvalues of the corresponding Gram matrix. Vectors satisfying $(\mathbf{x}|\mathbf{x}) = g_{\mu\nu} x^\mu x^\nu = 0$ are said to lie on the *light cone* and are referred to as *null vectors*. The light cone separates spacetime vectors into *spacelike* $((\mathbf{x}|\mathbf{x}) < 0)$ and *timelike* $((\mathbf{x}|\mathbf{x}) > 0)$ ones. The geodesic equations are formally identical to $(*)$, but with Levi-Civita Christoffel symbols pertaining to the Lorentzian rather than Riemannian metric.

(5) **e.** Explain the meaning of the general relativistic statement that "*null geodesics are invariant under conformal metric transforms*".
(*Hint:* Reconsider your answers under c1–c2 in the context of the stipulated Lorentzian metric.)

(50) **2.** CARTOGRAPHY

In this problem we model the earth's surface as the unit sphere $\mathbb{S} \subset \mathbb{E}$ embedded in Euclidean space $\mathbb{E} = \mathbb{R}^3$ by parametrizing the Cartesian coordinates $(x^1, x^2, x^3) = (x, y, z) \in \mathbb{R}^3$ of a point $\mathscr{P} \in \mathbb{S}$ in terms of polar angles $(u^1, u^2) = (\phi, \theta)$, as follows:

$$E : \mathbb{S} \to \mathbb{E} : (\phi, \theta) \mapsto (x = E^1(\phi, \theta), y = E^2(\phi, \theta), z = E^3(\phi, \theta)) : \begin{cases} E^1(\phi, \theta) &= \cos\phi\cos\theta \\ E^2(\phi, \theta) &= \sin\phi\cos\theta \\ E^3(\phi, \theta) &= \sin\theta\ , \end{cases}$$

with $(\phi, \theta) \in (-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. This mapping naturally induces *push-forward* and *pull-back* maps, $E_* : \mathrm{T}\mathbb{S} \to \mathrm{T}\mathbb{E} : \mathbf{v} \mapsto E_*\mathbf{v}$, respectively $E^* : \mathrm{T}^*\mathbb{E} \to \mathrm{T}^*\mathbb{S} : \boldsymbol{\omega} \mapsto E^*\boldsymbol{\omega}$, in which $\mathbf{v} = v^a\partial_a \in \mathrm{T}\mathbb{S}$ and $\boldsymbol{\omega} = \omega_i dx^i \in \mathrm{T}^*\mathbb{E}$. These are defined via linear extension through their actions on basis (co-)vectors:

$$E_*\partial_a = E^i_a\partial_i \quad \text{respectively} \quad E^*dx^i = E^i_a du^a \quad \text{with Jacobian} \quad E^i_a = \frac{\partial E^i}{\partial u^a}\,.$$

Notation. Summation convention applies to repeated indices. We use shorthands $\partial_a = \frac{\partial}{\partial u^a}$ and $\partial_i = \frac{\partial}{\partial x^i}$. Indices $a, b, c, \ldots = 1, 2$ from the beginning of the alphabet pertain to the 2-dimensional unit sphere $\mathbb{S}$, indices $i, j, k, \ldots = 1, 2, 3$ from the middle of the alphabet to the 3-dimensional Euclidean space $\mathbb{E}$.

(5) **a1.** Express each of the 1-forms E^*dx, E^*dy, and E^*dz in terms of $d\phi$ and $d\theta$.

(5) **a2.** Express each of the vectors $E_*\partial_\phi$, and $E_*\partial_\theta$ in terms of ∂_x, ∂_y, and ∂_z.

The Euclidean metric tensor on $\mathbb{E}$ is given by $\mathbf{h} = h_{ij}\, dx^i \otimes dx^j = dx \otimes dx + dy \otimes dy + dz \otimes dz$. This metric naturally induces a *pull-back metric* $\mathbf{g} = E^*\mathbf{h} = g_{ab}\, du^a \otimes du^b = (h_{ij} \circ E)\, E^*dx^i \otimes E^*dx^j$ on $\mathbb{S}$. The symbol $\circ$ denotes composition, i.e. $(h_{ij} \circ E)(\phi, \theta) = h_{ij}(E(\phi, \theta))$.

(5) **b1.** Show that, in general, $g_{ab}\, du^a \otimes du^b = \dfrac{\partial E^i}{\partial u^a}(h_{ij} \circ E)\dfrac{\partial E^j}{\partial u^b}\, du^a \otimes du^b$.

(5) **b2.** Show that, for the particular choice of the mapping E above, $\mathbf{g} = \cos^2\theta d\phi \otimes d\phi + d\theta \otimes d\theta$.

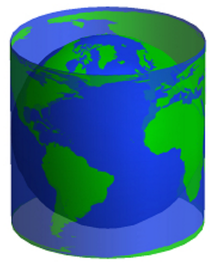
The Christoffel symbols on $\mathbb{S}$ in u-coordinates are given by $\Gamma^c_{ab} = \dfrac{1}{2}g^{c\ell}\,(\partial_a g_{\ell b} + \partial_b g_{a\ell} - \partial_\ell g_{ab})$.

($7\frac{1}{2}$) **c1.** Compute the following Christoffel symbols on $\mathbb{S}$: (i) $\Gamma^\phi_{\phi\theta}$, (ii) $\Gamma^\phi_{\theta\phi}$, and (iii) $\Gamma^\theta_{\phi\phi}$.

($2\frac{1}{2}$) **c2.** Pick *one* of the *remaining* symbols Γ^c_{ab} not considered in problem c1, and show that it vanishes.

(5) **d1.** Provide the geodesic equations for a parametrized geodesic on $\mathbb{S}$: $(\phi, \theta) = (\phi(t), \theta(t))$, with $t \in \mathbb{R}$.

(5) **d2.** Show that the equator is a geodesic, and argue why this implies that *all* great circles (intersections of planes with $\mathbb{S}$ through the earth's center) are geodesics.

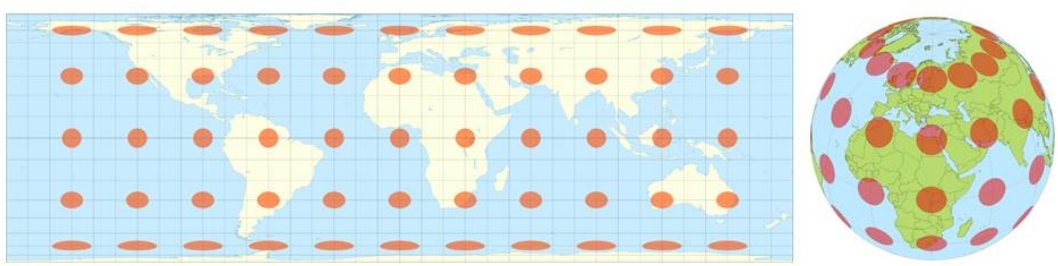


LAMBERT CYLINDRICAL EQUAL-AREA PROJECTION (JOHANN HEINRICH LAMBERT, 1728–1777).

The *Lambert cylindrical equal-area projection*

$$L:\mathbb{S}\rightarrow\mathbb{M}:(\phi,\theta)\mapsto(\xi=L^1(\phi,\theta),\eta=L^2(\phi,\theta)):\left\{\begin{array}{lcl} L^1(\phi,\theta) & = & \phi \\ L^2(\phi,\theta) & = & \sin\theta\,, \end{array}\right.$$

with $(\phi,\theta)\in(-\pi,\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}]$, provides a flat chart $\mathbb{M}$ of the earth's surface $\mathbb{S}$, cf. the figure.



TISSOT'S INDICATRICES OF DEFORMATION ARE USED TO VISUALIZE SURFACE DISTORTIONS DUE TO PROJECTION.

The unit volume form on an n-dimensional Riemannian space is given by $\boldsymbol{\epsilon}=\sqrt{g}\,dz^1\wedge\ldots\wedge dz^n$ in any coordinate basis $\{dz^1,\ldots,dz^n\}$, in which g denotes the determinant of the covariant metric tensor. Below we refer to a 2-dimensional volume form as an "area form".

(5) **e1.** Show that the unit area 2-form on the sphere $\mathbb{S}$ is given by $\boldsymbol{\epsilon}_{\mathbb{S}}=\cos\theta d\phi\wedge d\theta$.

We endow $\mathbb{M}$ with a 2-dimensional Euclidean structure with Cartesian coordinates $(\xi^1=\xi,\xi^2=\eta)$. In terms of these coordinates the metric on $\mathbb{M}$ takes the form $\boldsymbol{\eta}=\eta_{\mu\nu}d\xi^\mu\otimes d\xi^\nu=d\xi\otimes d\xi+d\eta\otimes d\eta$, with trivial metric determinant $\eta=1$.

(5) **e2.** Prove that Lambert cylindrical equal-area projection preserves areas, as follows. Show that the area form induced by pulling back the *unit* area form on the earth's map onto the earth's surface under L^*, i.e. $\boldsymbol{\omega}_{\mathbb{S}}=L^*\boldsymbol{\epsilon}_{\mathbb{M}}\stackrel{\text{def}}{=}\sqrt{\eta\circ L}\,L^*d\xi\wedge L^*d\eta$, is in fact the *unit* area form on $\mathbb{S}$, i.e. show that $\boldsymbol{\omega}_{\mathbb{S}}=\boldsymbol{\epsilon}_{\mathbb{S}}$.

THE END