EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0. Date: Tuesday April 7, 2020. Time: 09h00–12h00. Canvas online assignment.

READ THIS FIRST!

- Write your name and student identification number on each paper, and include it in the file name(s) of any scan you upload via Assignments in Canvas.
- The exam consists of 3 problems and a mandatory legal statement. Credits are indicated in the margin.
- Follow the legal statement instruction at the end of the exam before you scan and upload your results! *This requires you to include a written statement exactly as indicated in that clause.*
- You may consult the online draft notes “Tensor Calculus and Differential Geometry (2WAH0)” by Luc Florack to be found in the Files folder of Canvas. No other material or equipment may be used.
- The host/invigilator may ask you to temporarily share your screen *promptly without undue delay* at any given moment via a chat request directly to you. Your screen should display no other items than those permitted for this exam (online course notes, Zoom, Canvas).
- Keep your camera on and do not unmute yourself, as this may be distractive.
- You can chat with the host/invigilator (only) for urgent requests. Do not step away from the camera without such a request and before explicit approval. Use this option only if strictly necessary. In principle, only one short sanitary break will be allowed during the exam.
- The Einstein summation convention is in effect throughout this exam.

(20) 1. TENSOR CALCULUS MISCELLANY.

Prove the following conjectures, in which V is a vector space of dimension $\dim V \doteq n \geq 2$:

(5) **a.** A completely antisymmetric tensor $T : \underbrace{V \times \ldots \times V}_{n \text{ copies}} \rightarrow \mathbb{R}$ has exactly one degree of freedom.

A tensor is completely determined by its holor. Therefore, consider n arbitrary basis vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n} \in V$. By virtue of antisymmetry (exploited in $\star$) we may rearrange $T_{i_1 \ldots i_n} \doteq T(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n}) \stackrel{\star}{=} [i_1 \ldots i_n]\, T(\mathbf{e}_1, \ldots, \mathbf{e}_n) \doteq [i_1 \ldots i_n]\, T_{1,\ldots,n}$, so that the single number $k \doteq T_{1,\ldots,n} \in \mathbb{R}$ disambiguates the tensor completely.

(5) **b.** If the holor X_{ij} is antisymmetric, while the holor Y^{ij} is symmetric, then $X_{ij} Y^{ij} = 0$.

A dummy index manipulation ($\circ$), using the stipulated (anti)symmetry ($\star$), reveals $X_{ij} Y^{ij} \stackrel{\star}{=} X_{ij} Y^{ji} \stackrel{\star}{=} -X_{ji} Y^{ji} \stackrel{\circ}{=} -X_{ij} Y^{ij}$.

(5) **c.** A rank-3 holor cannot be decomposed into symmetric and antisymmetric parts: $T_{ijk} \neq T_{(ijk)} + T_{[ijk]}$.

This follows from a dimensionality argument:

- A general rank-3 holor has n^3 independent components.
- A symmetric rank-3 holor has $\binom{n+2}{3} = \frac{1}{6}n(n+1)(n+2)$ independent components.
- An antisymmetric rank-3 tensor has $\binom{n}{3} = \frac{1}{6}n(n-1)(n-2)$ independent components.
- The sum of the latter two numbers equals $\frac{1}{3}n(n^2+2) < n^3$, which can be proven by induction for $n \geq 2$ (or by observing $n < n^3$).

(5) **d.** If V has a Lorentzian inner product, the Hodge star of the unit volume form $\boldsymbol{\epsilon} \doteq \sqrt{|g|}\,\hat{\mathbf{e}}^1 \wedge \ldots \wedge \hat{\mathbf{e}}^n$ relative to a positively oriented basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ equals $*\boldsymbol{\epsilon} = -1$. Here $g \doteq \det g_{\bullet\bullet}$ denotes the determinant of the Gram matrix.

The $\boldsymbol{\epsilon}$ tensor is an n-form, so its Hodge star $*\boldsymbol{\epsilon}$ is a 0-form, i.e. a real number. Using $\sqrt{|g|}\,\hat{\mathbf{e}}^1 \wedge \ldots \wedge \hat{\mathbf{e}}^n = \epsilon_{i_1 \ldots i_n}\,\hat{\mathbf{e}}^{i_1} \otimes \ldots \otimes \hat{\mathbf{e}}^{i_n}$, with tensor holor $\epsilon_{i_1 \ldots i_n} = \sqrt{|g|}\,[i_1 \ldots i_n]$, we obtain

$$*\boldsymbol{\epsilon} = \frac{1}{n!}\epsilon_{i_1 \ldots i_n} g^{i_1 j_1} \ldots g^{i_n j_n} \epsilon_{j_1 \ldots j_n} = \frac{1}{n!}\sqrt{|g|}\,[i_1 \ldots i_n] g^{i_1 j_1} \ldots g^{i_n j_n} \sqrt{|g|}\,[j_1 \ldots j_n] = |g|\,\det g^{\bullet\bullet} = \frac{|g|}{g} = -1\,.$$

The last step follows from the fact that, by definition, the Gram matrix $g_{\bullet\bullet}$ of a Lorentzian inner product (and thus also its inverse $g^{\bullet\bullet}$) has one eigenvalue of opposite sign relative to the $n-1$ remaining eigenvalues, whence the product of eigenvalues, and thus $\det g^{\bullet\bullet}$, is negative.

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(30) 2. Lorentzian Inner Product.

We consider the vector space of real symmetric 2×2 matrices,

$$\mathbb{M}_2^{\text{sym}} \doteq \left\{ X = \begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix} \middle| \, x^{ab} = x^{ba} \in \mathbb{R} \right\},$$

and furnish it with a symmetric bilinear operator $(\_|\_) : \mathbb{M}_2^{\text{sym}} \times \mathbb{M}_2^{\text{sym}} \to \mathbb{R} : (X, Y) \mapsto (X|Y)$. Recall that real symmetric matrices have real eigenvalues. In order to define this bilinear operator we start out by defining a *pseudonorm* $\|\_\| : \mathbb{M}_2^{\text{sym}} \to \mathbb{R} : X \mapsto \|X\|$, stipulated to have the following property:

$$(\bullet) \qquad \|X\|^2 \doteq \frac{1}{2}(X|X)\,.$$

(5) **a.** Show that, for any $X, Y \in \mathbb{M}_2^{\text{sym}}$, $(X|Y)$ is uniquely determined in terms of this pseudonorm. [*Hint:* Expand $(X - Y|X - Y)$.]

Expanding $(X - Y|X - Y) = (X|X) - 2\,(X|Y) + (Y|Y)$ allows us to express $(X|Y)$ in terms of the pseudonorm already defined, viz.

$$(X|Y) = \frac{1}{2}(X|X) + \frac{1}{2}(Y|Y) - \frac{1}{2}(X - Y|X - Y) \doteq \|X\|^2 + \|Y\|^2 - \|X - Y\|^2\,.$$

Note that both bilinearity as well as symmetry of $(\_|\_)$ have been used in the expansion.

We stipulate the following instance for the pseudonorm (thus also disambiguating the bilinear map):

$$(\star) \qquad \|X\|^2 \doteq \det X\,,$$

and henceforth assume this to hold. By $\text{adj}Y$ we denote the adjugate (transposed cofactor) matrix of Y.

(5) **b1.** Use the definition of adj$Y$ to prove that if $Y \doteq \begin{pmatrix} y^{11} & y^{12} \\ y^{21} & y^{22} \end{pmatrix}$ then adj$Y = \begin{pmatrix} y^{22} & -y^{12} \\ -y^{21} & y^{11} \end{pmatrix}$.

This follows by a straightforward computation of the four elements $(\text{adj}Y)^{ab} = \partial \det Y / \partial y^{ba}$, using $\det Y = y^{11}y^{22} - y^{21}y^{12}$.

(5) **b2.** Recall (•), (⋆). Show, by explicit computation in terms of matrix elements, that

$$(X|Y) = \text{tr}\,(X\text{adj}Y)\,.$$

According to problem a and the above instance of the pseudonorm we have

$$(X|Y) \overset{\text{a}}{=} \|X\|^2 + \|Y\|^2 - \|X-Y\|^2 \doteq \det X + \det Y - \det(X-Y)\,.$$

A brute force computation of the r.h.s. in terms of matrix components, setting

$$X = \begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y^{11} & y^{12} \\ y^{21} & y^{22} \end{pmatrix}$$

with $x^{12} = x^{21}$ and $y^{12} = y^{21}$, yields, after a miraculous cancellation of terms,

$$\det X + \det Y - \det(X-Y) = x^{11}y^{22} + x^{22}y^{11} - 2x^{12}y^{12}\,.$$

On the other hand, using

$$\text{adj}Y = \begin{pmatrix} y^{22} & -y^{12} \\ -y^{21} & y^{11} \end{pmatrix}$$

we likewise find

$$\text{tr}\,(X\text{adj}Y) = x^{11}y^{22} + x^{22}y^{11} - 2x^{12}y^{12}\,.$$

These calculations confirm the conjectured identity $(X|Y) = \text{tr}\,(X\text{adj}Y)$.

(5) **c.** Show that $(\_|\_)$ is indeed symmetric bilinear. Would this be the case if we had considered $\mathbb{M}_n^{\text{sym}}$ instead of $\mathbb{M}_2^{\text{sym}}$, i.e. the vector space of real symmetric $n \times n$ matrices, with $n \neq 2$?

This follows from b2, since $(X|Y) = \text{tr}\,(X\text{adj}Y) = x^{11}y^{22} + x^{22}y^{11} - 2x^{12}y^{12}$ is clearly symmetric and bilinear. Note that this result relies explicitly on the fact that adjugation, $Z(Y) \doteq \text{adj}Y$, is linear in Y in $n=2$ dimensions, and therefore does *not* hold if $n \neq 2$.

We now interpret $x^A \doteq x^{ab} \in \mathbb{R}$ as the contravariant 'metarank'-one holor of the vector $X \in \mathbb{M}_2^{\text{sym}}$, in which each *index pair* (a,b) is treated as a single, so-called *multi-index* $A \doteq (a,b)$.

Definition. For $A \doteq (a,b)$ the covariant 'metarank'-one holor $\varepsilon_A \doteq \varepsilon_{ab}$ is defined as

$$\varepsilon_A \doteq \begin{cases} 1 & \text{if } A = (1,2) \\ -1 & \text{if } A = (2,1) \\ 0 & \text{otherwise.} \end{cases}$$

Expressed in multi-index convention we stipulate a covariant 'metarank'-two Gram matrix with holor g_{AB}, in which $A \doteq (a,b)$, $B \doteq (c,d)$, say, such that

$$(X|Y) \doteq g_{AB}\, x^A y^B\,.$$

(5) **d.** Express g_{AB}, with $A = (a,b)$, $B = (c,d)$, in terms of ε-symbols and original indices $a,b,c,d = 1,2$. [*Hint:* Problem a tells you that you may consider $(X|X) = g_{AB}x^Ax^B$ without loss of generality.]

We have, using the definition of the determinant $\det X \doteq \frac{1}{2}\varepsilon_{ac}\varepsilon_{bd}x^{ab}x^{cd}$,

$$g_{AB}\, x^A x^B \doteq (X|X) = 2\det X \doteq \varepsilon_{ac}\varepsilon_{bd}\, x^{ab}x^{cd} = \varepsilon_{ac}\varepsilon_{bd}x^A x^B \,.$$

Thus we must identify

$$g_{AB} = \varepsilon_{ac}\varepsilon_{bd} \,.$$

(Do not confuse the r.h.s. with the tensor product $\varepsilon_A\varepsilon_B$.) By problem a this implies $(X|Y) = g_{AB}\, x^A y^B$. As an aside, note that if we would agree on an index lowering convention in which $x^b_c \doteq \varepsilon_{ac}x^{ab}$, then we may also write $(X|Y) = x^b_c y^c_b$, which is just the standard inner product on the linear space of *traceless mixed* matrices, the natural counterparts of our symmetric contravariant matrices.

(5) **e.** Show that $(\_|\_)$ is a nondegenerate Lorentzian inner product.

Symmetry and bilinearity have been proven in c. Suppose furthermore that $X \in \mathbb{M}_2^{\text{sym}}$ is a nonzero matrix. Then at least one of the coefficients x^{ab} in the expression $(X|Y) = x^{11}y^{22} + x^{22}y^{11} - 2x^{12}y^{12}$ must be nozero. As a result there exists a nonzero matrix $Y \in \mathbb{M}_2^{\text{sym}}$ such that the r.h.s. does not vanish, so that the inner product is of pseudo-Riemannian type. In fact, from $(X|X) = 2\det X$ it follows that $(X|X)$ may be positive, zero, or negative, depending on the matrix X, and since $\dim \mathbb{M}_2^{\text{sym}} = 3$, its signature must be Lorentzian.

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(50) 3. Retino-cortical Map.

We consider a geometric model of the retino-cortical map, the neural pathway that transfers optical input signals from retina to cortical surface (both viewed as compact subsets of $\mathbb{R}^2$). This map is topology preserving, but distortive in the sense that it has an eccentricity dependent effect on area measures.

Notation:

- $(x, y) \in \mathbb{R}^2$: Cartesian coordinates in 2-dimensional Euclidean space;
- $(dx, dy) \in \mathrm{T}^*\mathbb{R}^2{}_{(x,y)}$: coordinate 1-forms;
- $a > 0$: size parameter (a physical constant);
- $(r, \phi) \in \mathbb{R}_0^+ \times [\phi_0, \phi_0 + 2\pi)$: polar coordinates, with $(x, y) = (r\cos\phi, r\sin\phi)$, $\phi_0 \in \mathbb{R}$ fixed.

Consider the following two non-exact one-form fields, in which $a > 0$ is a parameter:

$$\bar{d}\xi = \frac{dx}{\sqrt{x^2+y^2+a^2}} \quad \text{and} \quad \bar{d}\eta = \frac{dy}{\sqrt{x^2+y^2+a^2}} \,,$$

The barred symbol $\bar{d}$ stresses the non-exact nature of the one-forms ($d\bar{d} \neq 0$).

Let $v = v^x\partial_x + v^y\partial_y$ be an arbitrary vector field, with component functions $v^x, v^y \in C^\infty(\mathbb{R}^2)$.

(5) **a.** Compute $\bar{d}\xi(v)$ and $\bar{d}\eta(v)$ as functions of x and y.

Using linearity and duality, $dx(\partial_x) = dy(\partial_y) = 1$, $dx(\partial_y) = dy(\partial_x) = 0$, in the expressions for $\bar{d}\xi(v)$ and $\bar{d}\eta(v)$ yields

$$\bar{d}\xi(v)(x,y) = \frac{v^x(x,y)}{\sqrt{x^2+y^2+a^2}} \quad \text{and} \quad \bar{d}\eta(v)(x,y) = \frac{v^y(x,y)}{\sqrt{x^2+y^2+a^2}} \,.$$

(5) **b.** Compute $d\,đ\xi$ and $d\,đ\eta$ in terms of x, y, dx, dy to reveal the non-exact nature of $đ\xi$ and $đ\eta$.

Using $dd \doteq d \circ d = 0$ and antisymmetry of the $\wedge$-product we find

$$d\,đ\xi = \frac{y\,dx \wedge dy}{(x^2+y^2+a^2)^{\frac{3}{2}}} \quad \text{and} \quad d\,đ\eta = \frac{-x\,dx \wedge dy}{(x^2+y^2+a^2)^{\frac{3}{2}}}$$

The latter follows by symmetry $(x, y) \leftrightarrow (y, x)$ from the former. Apparently the one-forms $đ\xi$ and $đ\eta$ are not closed and therefore non-exact.

We stipulate a Riemannian structure, with Gram matrix g_{ij} still to be determined, such that

$$\varepsilon = \sqrt{g}\,dx \wedge dy \doteq đ\xi \wedge đ\eta\,.$$

Here ε denotes the Riemannian unit area form, or Levi-Civita tensor, and $g \doteq \det(g_{ij})_{1\leq i,j\leq 2}$.

(5) **c.** Compute $\sqrt{g}$ as a function of x and y.

Using bilinearity and the definitions of $đ\xi$ and $đ\eta$ in terms of x, y, dx, dy we obtain

$$đ\xi \wedge đ\eta = \frac{dx \wedge dy}{x^2+y^2+a^2} \doteq \sqrt{g}\,dx \wedge dy\,,$$

from which we read off the factor $\sqrt{g(x,y)} = \dfrac{1}{x^2+y^2+a^2}$

($2\frac{1}{2}$) **d1.** Compute the area $\delta V(r) \doteq \displaystyle\iint_{\delta\Omega(r)} đ\xi \wedge đ\eta$ of the narrow ring $\delta\Omega(r) : r < \sqrt{x^2+y^2} < r+\delta r$, for some constants $r>0$ and $\delta r>0$.

Expressing the area form in (x, y)-coordinates as in c, and switching to polar coordinates $(x, y) = (\rho\cos\phi, \rho\sin\phi)$, yields

$$\delta V(r) = \frac{2\pi r\delta r}{r^2+a^2}\,.$$

($2\frac{1}{2}$) **d2.** Infer from this the retino-cortical magnification $V'(r) = \dfrac{dV(r)}{dr}$ and show that its integral $V(r)$, subject to the condition $V(0)=0$, is given by

$$V(r) = \pi \ln\left(1 + \left(\frac{r}{a}\right)^2\right)\,.$$

Taking the limit $\delta r \to 0$ in d1 produces

$$V'(r) = \frac{2\pi r}{r^2+a^2}\,.$$

Integration, subject to $V(0)=0$, readily yields

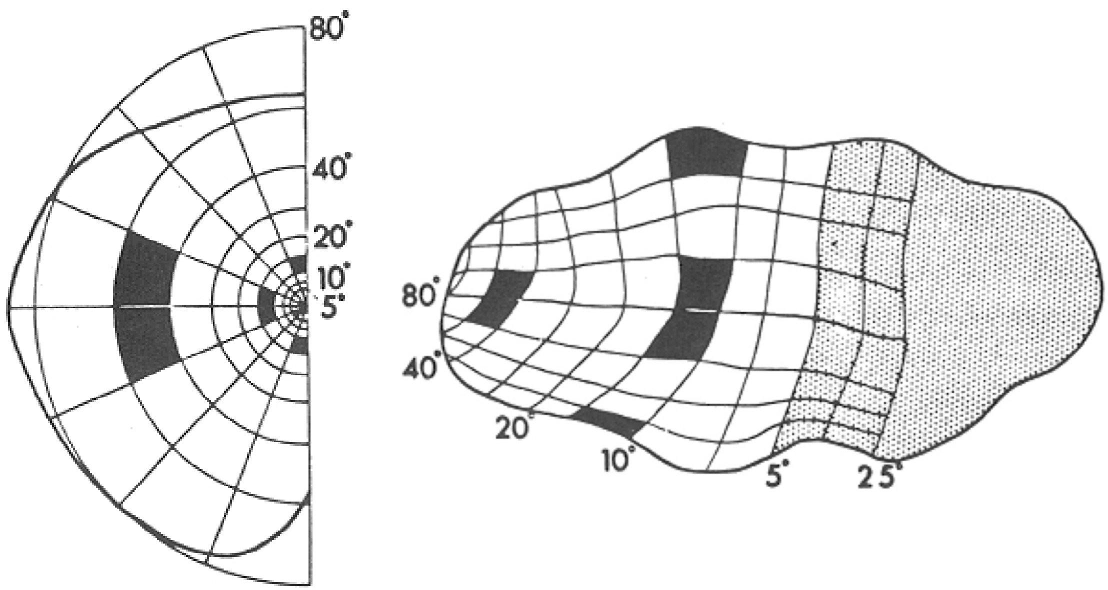
$$V(r) = \int_0^r V'(\rho)\,d\rho = \pi \ln\left(1 + \left(\frac{r}{a}\right)^2\right)\,.$$

We assume the compact retinal domain to be the disc $\Omega : 0 \leq r \leq R$ for some radius $R \gg a$.

$(2\frac{1}{2})$ **d3.** Compute the radius $r_{1/2}$ of the foveal region $\Omega_{1/2} : 0 \le r \le r_{1/2}$ that claims half of the cortical area, i.e. such that $V(r_{1/2}) \doteq \frac{1}{2} V(R)$.
[*Hint:* You may approximate $r_{1/2}$ to lowest order using $R \gg a$.]

Solving the equation $V(r_{1/2}) = \frac{1}{2}V(R)$ for $r_{1/2}$, using $R \gg a$, yields $r_{1/2} \approx \sqrt{aR}$, i.e. the geometric average of a and R.

Eccentricity is a measure for distance relative to the foveal centre, conventionally expressed in degrees of visual field, which is proportional to the radial coordinate r. Connolly and Van Essen claim that the central 5 degrees of visual field claim about 40% of the visual cortex (shaded area, cf. the illustration).



MAPPING OF THE VISUAL FIELD FROM RETINA (LEFT) TO STRIATE CORTEX (RIGHT) IN MONKEY ACCORDING TO CONNOLLY AND VAN ESSEN (© MICHAEL CONNOLLY AND DAVID VAN ESSEN, J. COMP. NEUROL. 226:4, 544–564,1984). THE REPRESENTATION OF THE CENTRAL 5 DEGREES OF THE RETINAL VISUAL FIELD OCCUPIES ABOUT 40% OF THE CORTEX (SHADED AREA ON THE RIGHT).

IN THE GEOMETRIC MODEL BY FLORACK (AXIOMS 3:1, 70–81, 2014) THE AREA FORM $\varepsilon = d\!\!\bar{}\,\xi \wedge d\!\!\bar{}\,\eta$ REPRESENTS THE SURFACE ELEMENT ON THE STRIATE CORTEX RETINOTOPICALLY CONNECTED TO A UNIT SURFACE ELEMENT $dx \wedge dy$ ON THE RETINA.

Based on the figure we consider the radius r_{5° of a disc corresponding to the central 5 degrees of visual field, taking the full visual field to have an eccentricity of approximately 80 degrees, so that $r_{80^\circ} \doteq R$. Introducing dimensionless coordinates we set $r_{5^\circ} \doteq \alpha R$, with $\alpha = r_{5^\circ}/R = r_{5^\circ}/r_{80^\circ} = 5/80$. We furthermore introduce the dimensionless parameter β via $V(\alpha R) \doteq \beta V(R)$.

$(2\frac{1}{2})$ **d4.**. Verify whether Connolly and Van Essen's observation, viz. that $\beta \approx 40\%$, is indeed consistent with our geometric model if we choose $\alpha = 5/80$. Use the approximation $R/a \gg 1$.

Set $t \doteq R/a \gg 1$ for convenience. Solving $V(\alpha R)/V(R) = \beta$ for β, given α, yields the equation $\ln(1 + \alpha^2 t^2) = \beta \ln(1 + t^2)$, whence

$1 + \alpha^2 t^2 = (1 + t^2)^\beta \approx 1 + \beta t^2$, from which we conclude that $\beta \approx \alpha^2 = (5/80)^2 \approx 39\%$, remarkably close to Connolly and Van Essen's observation.

(5) **e.** Compute the area form ε in terms of the 'canonical' coordinates (p, q), defined relative to the (slightly modified) polar coordinates (r, ϕ), with $0 \le r \le R$ and $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ (one visual hemisphere) by

$$\begin{cases} p &= \operatorname{arcsinh}\left(\dfrac{r}{a}\right) \\ q &= \dfrac{r\phi}{\sqrt{r^2 + a^2}} \,. \end{cases}$$

[*Hint:* You may use the identity $\operatorname{arcsinh}(t) = \ln(t + \sqrt{1 + t^2})$.]

We have $\varepsilon = dp \wedge dq$, which explains the attribute 'canonical'.

We furthermore stipulate a *retino-cortical metric* of the form (with $1 \le i, j \le 2$)

$$g_{ij}\, dx^i \otimes dx^j \doteq d\!\!\!{}^-\xi \otimes d\!\!\!{}^-\xi + d\!\!\!{}^-\eta \otimes d\!\!\!{}^-\eta \,.$$

(5) **f.** Compute g_{ij} as a function of x and y, and show that it is consistent with your answer under c.

Using once again the definitions of $d\!\!\!{}^-\xi$ and $d\!\!\!{}^-\eta$ in terms of x, y, dx, dy we obtain

$$\begin{bmatrix} g_{xx}(x,y) & g_{x,y}(x,y) \\ g_{yx}(x,y) & g_{yy}(x,y) \end{bmatrix} = \begin{bmatrix} \frac{1}{x^2+y^2+a^2} & 0 \\ 0 & \frac{1}{x^2+y^2+a^2} \end{bmatrix} .$$

Clearly $g(x,y) = \dfrac{1}{(x^2 + y^2 + a^2)^2}$ is consistent with c.

Lemma. In dimension 2 the covariant Riemann tensor takes the form $R_{ijk\ell} = K(g_{ik}g_{j\ell} - g_{i\ell}g_{jk})$ for some scalar K.

Definition. The Ricci tensor is defined by its holor $R_{ij} \doteq g^{k\ell} R_{ki\ell j}$. The Ricci scalar is $R \doteq g^{ij} R_{ij}$.

($2\frac{1}{2}$) **g1.** Show that the Ricci tensor is given by $R_{ij} = K\, g_{ij}$.

Taking the trace as indicated we have $R_{ij} \doteq g^{k\ell} R_{ki\ell j} = K\, g^{k\ell}(g_{k\ell} g_{ij} - g_{kj} g_{i\ell}) = K(2g_{ij} - g_{ij}) = K g_{ij}$. Note the factor 2 arising from $g^{k\ell} g_{k\ell} = n = 2$.

($2\frac{1}{2}$) **g2.** Show that $K = \dfrac{1}{2} R$, in which R is the Ricci scalar.

Contracting indices of the Ricci tensor to obtain the Ricci scalar yields, using the result of g1, $R \doteq g^{ij} R_{ij} = K g^{ij} g_{ij} = 2K$, where we have used once again the observation $g^{ij} g_{ij} = n = 2$.

(5) **h.** Show that if two metric tensors are related according to $g_{ij} = e^{2\alpha}\, h_{ij}$, in which $\alpha \in C^\infty(\mathbb{R}^2)$ is some scalar field, then their Christoffel symbols of the second kind, Γ^k_{ij} and H^k_{ij} say, are related by

$$\Gamma^k_{ij} = H^k_{ij} + \delta^k_i \partial_j \alpha + \delta^k_j \partial_i \alpha - h_{ij} \partial^k \alpha \,.$$

This follows straightforwardly from the formula of the Levi-Civita connection symbols Γ^k_{ij} in terms of g_{ij} and $\partial_k g_{ij}$, using the product rule for differentiation and setting $g_{ij} = e^{2\alpha}\, h_{ij}$. (The symbols H^k_{ij} are clearly those corresponding to h_{ij}.)

(5) **i.** Compute the Ricci scalar $R(x, y)$ for the retino-cortical metric $g_{ij}(x, y)$ and show that it is bounded.

Apparently the covariant Riemann tensor has only one independent component, which we may take to be R_{xyxy}, so that, using the fact that $g_{xy} = g_{yx} = 0$, we have $R\, g_{xx}\, g_{yy} = 2R_{xyxy}$. Evaluating R_{xyxy} in terms of the Γ-symbols from problem h, with $H^k_{ij} = 0$ (Euclidean space, Cartesian coordinates) and $e^{2\alpha(x,y)} = \frac{1}{x^2+y^2+a^2}$, yields

$$R(x, y) = \frac{4a^2}{x^2 + y^2 + a^2}\,.$$

Clearly we have $0 \leq R(x, y) \leq R(0, 0) = 4$.

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By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

Text to be copied (with optional remarks):

✍ *I made this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.*

Remarks from the student: . . . .

THE END