EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0. Date: Tuesday April 9, 2019. Time: 09h00–12h00. Place: Atlas 8.340.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. Credits are indicated in the margin.
- Items marked with ✍ may be relatively time consuming. It is safe to postpone them until the end.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No other material or equipment may be used.
- The Einstein summation convention is in effect throughout this exam.

(**25**) **1. TENSOR CALCULUS MISCELLANY.**

We consider a 2-dimensional real positive definite inner product space V with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. The metric tensor is given by $\mathbf{g} = g_{ij}\,\hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}^j$.

Let $A^{\text{cov}} = A_{ij}\,\hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}^j$ be a real symmetric covariant tensor relative to the dual basis $\{\hat{\mathbf{e}}^1, \hat{\mathbf{e}}^2\}$, and $A^{\text{mix}} = A^i_j\,\mathbf{e}_i \otimes \hat{\mathbf{e}}^j$ the associated mixed tensor with $A^i_j = g^{ik}A_{kj}$, with $g^{ik}g_{kj} = \delta^i_j$.

The 2×2 matrix $\mathbf{A}^{\text{cov}}$ is defined in terms of the real-valued holor A_{ij} of A^{cov}, viz.

$$\mathbf{A}^{\text{cov}} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \text{with } A_{21} = A_{12}.$$

The 2×2 matrix $\mathbf{A}^{\text{mix}}$ is defined in terms of the holor A^i_j of $A^{\text{mix}} = A^i_j\,\mathbf{e}_i \otimes \hat{\mathbf{e}}^j$, viz.

$$\mathbf{A}^{\text{mix}} = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix}.$$

(5) **a.** Is the matrix $\mathbf{A}^{\text{mix}}$ symmetric?

Typically not. We have $A^i_j = g^{ik}A_{kj}$, so $A^1_2 = g^{11}A_{12} + g^{12}A_{22} \neq g^{21}A_{11} + g^{22}A_{21} = g^{12}A_{11} + g^{22}A_{12} = A^2_1$, unless the Gram matrix satisfies additional constraints beyond symmetry (notably $g_{11} = g_{22}$ and $g_{12} = g_{21} = 0$). Note that we have fully exploited symmetry of $\mathbf{g}$ and $\mathbf{A}^{\text{cov}}$ by eliminating the dependent coefficients g_{21} and A_{21} in favor of g_{12} and A_{12}.

Below we consider the generic situation in which the matrix $\mathbf{A}^{\text{cov}}$ is nonsingular, so that $\det \mathbf{A}^{\text{cov}} \neq 0$.

(5) **b.** Does this imply that the matrix $\mathbf{A}^{\text{mix}}$ is nonsingular?

Yes. Writing $\det g_{ij}$, $\det g^{ij}$, $\det A_{ij}$ and $\det A^i_j$ for $\det \mathbf{g}$, $\det(\mathbf{g}^{-1})$, $\det \mathbf{A}^{\text{cov}}$ and $\det \mathbf{A}^{\text{mix}}$ by abuse of notation, we have $\det A^i_j = \det(g^{ik} A_{kj}) = \det g^{ik} \det A_{kj} = \det(\mathbf{g}^{-1}) \det \mathbf{A}^{\text{cov}} = (\det \mathbf{g})^{-1} \det \mathbf{A}^{\text{cov}} \neq 0$.

Definition. An eigenvector $v = v^i \mathbf{e}_i \in V$ of A^{mix} is a nonzero vector satisfying $\mathbf{A}^{\text{mix}}\, \mathbf{v} = \lambda \mathbf{v}$ for some eigenvalue $\lambda \in \mathbb{C}$, in which $\mathbf{v} \in \mathbb{C}^2$ is the column vector defined in terms of the holor v^i of $\mathbf{v}$, i.e.

$$\mathbf{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} .$$

Thus in the sense of classical matrix theory the column vector $\mathbf{v} \in \mathbb{C}^2$ is an eigenvector of the matrix $\mathbf{A}^{\text{mix}}$ with eigenvalue $\lambda \in \mathbb{C}$.

(5) **c.** Show that, if $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A}^{\text{mix}}$, then $\det(\mathbf{A}^{\text{mix}} - \lambda \mathbf{I}) = 0$, in which $\mathbf{I}$ is the 2×2 identity matrix.

The system $\mathbf{A}^{\text{mix}}\, \mathbf{v} = \lambda \mathbf{v}$ is equivalent to $(\mathbf{A}^{\text{mix}} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0} \in \mathbb{C}^2$. Since $\mathbf{v} \neq \mathbf{0}$ this implies that the matrix $\mathbf{A}^{\text{mix}} - \lambda \mathbf{I}$ must be singular, whence $\det(\mathbf{A}^{\text{mix}} - \lambda \mathbf{I}) = 0$.

Definition. The trace $\text{tr}\, \mathbf{A}^{\text{mix}}$ is the sum of the diagonal elements of $\mathbf{A}^{\text{mix}}$, i.e. $\text{tr}\, \mathbf{A}^{\text{mix}} = A^i_i$.

(5) **d.** Show that, in our 2-dimensional case and for general $\lambda \in \mathbb{C}$ (not necessarily an eigenvalue),

$$\det(\mathbf{A}^{\text{mix}} - \lambda \mathbf{I}) = \lambda^2 - \text{tr}\, \mathbf{A}^{\text{mix}}\, \lambda + \det \mathbf{A}^{\text{mix}} .$$

[*Hint:* Recall $\delta^{k\ell}_{ij} = [ij][k\ell] = \delta^k_i \delta^\ell_j - \delta^k_j \delta^\ell_i$.]

Using the definition of the determinant for dimension $n = 2$ we have

$$\det(\mathbf{A}^{\text{mix}} - \lambda \mathbf{I}) = \frac{1}{2} \delta^{k\ell}_{ij} (A^i_k - \lambda \delta^i_k)(A^j_\ell - \lambda \delta^j_\ell) = \frac{1}{2} \delta^{k\ell}_{ij} (A^i_k A^j_\ell - \lambda A^i_k \delta^j_\ell - \lambda A^j_\ell \delta^i_k + \lambda^2 \delta^i_k \delta^j_\ell) .$$

Working out the parentheses we recognize as the first term $\frac{1}{2} \delta^{k\ell}_{ij} A^i_k A^j_\ell = \det A^i_j$ (with the same remark on the r.h.s. notation as in the solution to problem b). For the remaining three terms it is helpful to rewrite

$$\delta^{k\ell}_{ij} = \delta^k_i \delta^\ell_j - \delta^k_j \delta^\ell_i .$$

As a result we find for the second and third term $-\frac{1}{2} \lambda \delta^{k\ell}_{ij} (A^i_k \delta^j_\ell + A^j_\ell \delta^i_k) = -\frac{1}{2} \lambda (\delta^k_i \delta^\ell_j - \delta^k_j \delta^\ell_i)(A^i_k \delta^j_\ell + A^j_\ell \delta^i_k) = -\lambda A^i_i$, and for the last term $\frac{1}{2} \lambda^2 \delta^{k\ell}_{ij} \delta^i_k \delta^j_\ell = \frac{1}{2} \lambda^2 (\delta^k_i \delta^\ell_j - \delta^k_j \delta^\ell_i) \delta^i_k \delta^j_\ell = \lambda^2$. In these computations we have used $\delta^i_i = n = 2$. Putting all terms together yields the stated identity.

Definition. We define the set S of scalar invariants associated with the tensor A^{mix} as follows:

$$S = \{ t_k \in \mathbb{R} \mid t_k \doteq \text{tr}\, (\underbrace{\mathbf{A}^{\text{mix}} \ldots \mathbf{A}^{\text{mix}}}_{k\text{-th matrix power}}) = \delta^{\ell_{k+1}}_{\ell_1} A^{\ell_1}_{\ell_2} A^{\ell_2}_{\ell_3} \ldots A^{\ell_k}_{\ell_{k+1}} \}_{k \in \mathbb{N}} .$$

(5) **e.** Prove that each t_k is indeed invariant in the sense of being independent of the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of V.

Let $\mathbf{e}_i = T^\ell_i \mathbf{f}_\ell$ be a basis transformation. Then $\hat{\mathbf{e}}^i = S^i_k \hat{\mathbf{f}}^k$, with $T^i_k S^k_j = \delta^i_j$, and $A^{\text{mix}} = A^i_j\, \mathbf{e}_i \otimes \hat{\mathbf{e}}^j = A^i_j\, S^j_k\, T^\ell_i\, \mathbf{f}_\ell \otimes \hat{\mathbf{f}}^k = \overline{A}^\ell_k\, \mathbf{f}_\ell \otimes \hat{\mathbf{f}}^k$. We recognize the holor of A^{mix} relative to the new basis as $\overline{A}^\ell_k = A^i_j\, S^j_k\, T^\ell_i$. Inserting this in the formula for $\bar{t}_k = \overline{A}^{\ell_{k+1}}_{\ell_2} \overline{A}^{\ell_2}_{\ell_3} \ldots \overline{A}^{\ell_k}_{\ell_{k+1}} =$

$A^{j_1}_{i_1} A^{j_2}_{i_2} \dots A^{j_k}_{i_k} S^{i_1}_{\ell_2} S^{i_2}_{\ell_3} \dots S^{i_k}_{\ell_{k+1}} T^{\ell_{k+1}}_{j_1} T^{\ell_2}_{j_2} \dots T^{\ell_k}_{j_k} = A^{j_1}_{i_1} A^{j_2}_{i_2} \dots A^{j_k}_{i_k} \delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \dots \delta^{i_k}_{j_1} = A^{j_1}_{j_2} A^{j_2}_{j_2} \dots A^{j_k}_{j_1} = t_k$. A less cumbersome proof relies on induction, in which case it suffices to prove that $\overline{X}^i_i = X^i_i$ is invariant for any mixed tensor X^i_j with invariant components.

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(25) 2. Area Two-Forms.

The wedge product of two covectors $\hat{\mathbf{v}}, \hat{\mathbf{w}} \in V^*$, i.e. $\hat{\mathbf{v}} \wedge \hat{\mathbf{w}}$, may be interpreted as an antisymmetric cotensor of rank 2, i.e. a bilinear map $\hat{\mathbf{v}} \wedge \hat{\mathbf{w}} : V \times V \to \mathbb{R} : (\mathbf{a}, \mathbf{b}) \mapsto (\hat{\mathbf{v}} \wedge \hat{\mathbf{w}})(\mathbf{a}, \mathbf{b})$, given by

$$(\hat{\mathbf{v}} \wedge \hat{\mathbf{w}})(\mathbf{a}, \mathbf{b}) = \det \begin{pmatrix} \langle \hat{\mathbf{v}}, \mathbf{a} \rangle & \langle \hat{\mathbf{v}}, \mathbf{b} \rangle \\ \langle \hat{\mathbf{w}}, \mathbf{a} \rangle & \langle \hat{\mathbf{w}}, \mathbf{b} \rangle \end{pmatrix} .$$

That is, $\hat{\mathbf{v}} \wedge \hat{\mathbf{w}} \in V^* \otimes_A V^* \doteq \bigwedge_2(V)$.

Similarly we may define the wedge product of two vectors, $\mathbf{a}, \mathbf{b} \in V$, i.e. $\mathbf{a} \wedge \mathbf{b}$, as an antisymmetric contratensor of rank 2, i.e. a bilinear map $\mathbf{a} \wedge \mathbf{b} : V^* \times V^* \to \mathbb{R} : (\hat{\mathbf{v}}, \hat{\mathbf{w}}) \mapsto (\mathbf{a} \wedge \mathbf{b})(\hat{\mathbf{v}}, \hat{\mathbf{w}})$, given by

$$(\mathbf{a} \wedge \mathbf{b})(\hat{\mathbf{v}}, \hat{\mathbf{w}}) = \det \begin{pmatrix} \langle \hat{\mathbf{v}}, \mathbf{a} \rangle & \langle \hat{\mathbf{w}}, \mathbf{a} \rangle \\ \langle \hat{\mathbf{v}}, \mathbf{b} \rangle & \langle \hat{\mathbf{w}}, \mathbf{b} \rangle \end{pmatrix} .$$

That is, $\mathbf{a} \wedge \mathbf{b} \in V \otimes_A V \doteq \bigwedge^2(V)$.

(10) **a.** Prove that $(\hat{\mathbf{v}} \wedge \hat{\mathbf{w}})(\mathbf{a}, \mathbf{b})$ actually depends only on $\mathbf{a} \wedge \mathbf{b}$, rather than independently on $\mathbf{a}$ and $\mathbf{b}$.

From the fact that the determinant of a matrix equals that of the transposed matrix it follows that $(\hat{\mathbf{v}} \wedge \hat{\mathbf{w}})(\mathbf{a}, \mathbf{b}) = (\mathbf{a} \wedge \mathbf{b})(\hat{\mathbf{v}}, \hat{\mathbf{w}})$. This shows that the independent variable for $\hat{\mathbf{v}} \wedge \hat{\mathbf{w}}$ is actually the single two-form $\mathbf{a} \wedge \mathbf{b} \in V \otimes_A V$ rather than the vector pair $(\mathbf{a}, \mathbf{b}) \in V \times V$.

(10) **b.** Explain the meaning of the identification $V^* \otimes_A V^* = (V \otimes_A V)^*$.

The l.h.s. denotes the linear space of antisymmetric *bilinear* mappings of type $V \times V \to \mathbb{R}$, as formulated above. The r.h.s. denotes the linear space of linear mappings of type $V \otimes_A V \to \mathbb{R}$. The statement $\phi \in V^* \otimes_A V^*$ says that $\phi : V \times V \to \mathbb{R}$ is an antisymmetric bivariate function of *two* vector arguments, $\mathbf{a}$ and $\mathbf{b}$ say. The statement $\phi \in (V \otimes_A V)^*$ says that $\phi : V \otimes_A V \to \mathbb{R}$ is a linear function of a single variable, viz. a vector in the form of an antisymmetric contratensor of rank 2, i.c. $\mathbf{a} \wedge \mathbf{b}$. The identification is justified by the observation in problem a, and boils down to the identification of *equivalence classes* of vector pairs $(\mathbf{a}, \mathbf{b}) \sim (\mathbf{a}', \mathbf{b}')$ for which $\mathbf{a} \wedge \mathbf{b} = \mathbf{a}' \wedge \mathbf{b}'$.

We consider a linear transformation $L : V \times V \to V \times V : (\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{a}', \mathbf{b}') = L(\mathbf{a}, \mathbf{b})$, given by

$$\begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix} = \mathbf{L} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \quad \text{in which } \mathbf{L} \text{ is a } 2 \times 2 \text{ matrix.}$$

(5) **c.** Under which conditions on $\mathbf{L}$ do we have $\mathbf{a} \wedge \mathbf{b} = \mathbf{a}' \wedge \mathbf{b}'$?

Set

$$\mathbf{L} = \begin{pmatrix} \lambda & \mu \\ \nu & \rho \end{pmatrix} .$$

Assuming $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors (otherwise no conditions need to be imposed on $\mathbf{L}$), we have $\mathbf{a}' \wedge \mathbf{b}' = (\lambda \mathbf{a} \wedge \mu \mathbf{b}) \wedge (\nu \mathbf{a} \wedge \rho \mathbf{b}) = (\lambda\rho - \mu\nu)\, \mathbf{a} \wedge \mathbf{b} = \det \mathbf{L}\, \mathbf{a} \wedge \mathbf{b}$, whence $\det \mathbf{L} = 1$.

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(50) 3. Newtonian Limit of Einstein Gravity.

Summation convention. In this problem we distinguish between two types of indices. Latin, or *spatial indices* $i, j, k, \ldots$ take values in the range $\{1, 2, 3\}$. Greek, or *spatiotemporal indices* $\mu, \nu, \rho, \ldots$ take values in the range $\{0, 1, 2, 3\}$. The summation convention for identical pairs of indices is in effect throughout this problem, with this tacit assumption on index range.

Gravity according to Newton. Newton regards spacetime as the product space of a flat Euclidean 3-space and 1-dimensional 'absolute time'. In 3-space inertial coordinates $(x^1, x^2, x^3) \doteq (x, y, z)$ exist such that the metric holor is $\eta_{ij} \doteq (\partial_i|\partial_j) = \delta_{ij}$ relative to the induced coordinate basis $\{\partial_i\}_{i=1,2,3}$. According to Newton, absolute time t progresses at a constant pace independently of any observer.

Kinematics. In Newtonian gravity one explains the response of matter to gravity in terms of a scalar gravitational potential field Φ entering Newton's second law of motion:

$$\frac{d^2x^i}{dt^2} = -\eta^{ij}\partial_j\Phi\,. \tag{1}$$

The classical 3-velocity has components $v^i \doteq dx^i/dt$ ($i = 1, 2, 3$). The r.h.s. represents the classical gravitational force field $\mathrm{g}^i = -\eta^{ij}\partial_j\Phi$ experienced by a test particle.

Dynamics. In turn, Φ originates from the spatial distribution of matter via the Poisson equation:

$$\Delta\Phi = 4\pi G\rho\,. \tag{2}$$

Here $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}$ (in Cartesian coordinates) is the Laplacian operator, ρ denotes mass density and $G \approx 6.674\,10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$ is the universal gravitational constant.

Gravity according to Einstein. According to Einstein, on the other hand, the gravitational potential is a symmetric rank-2 covariant tensor field identified with the Lorentzian spacetime metric $g_{\mu\nu} = (\partial_\mu|\partial_\nu)$ with signature $(-,+,+,+)$ in 4-dimensional spacetime relative to a coordinate basis $\{\partial_\mu\}_{\mu=0,1,2,3}$ induced by arbitrary coordinates $(x^0, x^1, x^2, x^3) \doteq (ct, x, y, z)$, with $c \doteq 299\,792\,458\,\mathrm{ms^{-1}}$.

Kinematics. The relativistic counterpart of (1) takes the form of a geodesic equation dictated by the metric $g_{\mu\nu}$ via the Levi-Civita connection:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0\,. \tag{3}$$

The observer independent affine parameter s is related to so-called 'proper time' τ via $s \doteq c\tau$. The vector with coefficients $V^\mu \doteq dx^\mu/d\tau$ is the (relativistic) 4-velocity, so we may equivalently write

$$\frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}V^\nu V^\rho = 0\,. \tag{3}$$

Its relation to the classical 3-velocity $v^i = dx^i/dt$ ($i = 1, 2, 3$) is given by $(V^0; V^i) = \gamma(v)(c; v^i)$, with time dilation factor

$$\gamma(v) = \frac{dct}{ds} = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}\,,$$

depending on the particle's Euclidean 3-velocity magnitude v. Note that, despite its extra dimension, the 4-velocity V^μ has the same three degrees of freedom as the 3-velocity v^i.

Dynamics. The counterpart of (2) is the Einstein field equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{4}$$

in which $T_{\mu\nu}$ is the stress-energy tensor. The left hand side of (4) is known as the Einstein tensor. We henceforth consider a stationary matter-energy distribution, for which $T_{00} = \rho\, c^2$, in which ρ denotes mass density, and $T_{\mu\nu} = 0$ otherwise.

(5) **a.** Assuming the holor $g_{\mu\nu}$ of the metric $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$ to be dimensionless, determine the physical units of the tensors listed below. Write the relevant dimensions in terms of monomials in L (*length*), T (*time*), M (*mass*), or use corresponding SI units m (*meter*), s (*second*), kg (*kilogram*).

[*Example:* G has dimension $\mathrm{L^3 M^{-1} T^{-2}}$ (SI: $\mathrm{m^3 kg^{-1} s^{-2}}$); $\|u\|^2 \doteq \mathbf{g}(u,u)$ has dimension L^2 (SI: m^2) if u is a spatiotemporal displacement vector, etc.]

a1. Riemann curvature tensor $R^\rho_{\mu\sigma\nu}$;

a2. Ricci curvature tensor $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$;

a3. Ricci curvature scalar $R = g^{\mu\nu} R_{\mu\nu}$;

a4. Stress-energy tensor $T_{\mu\nu}$.

Since $g_{\mu\nu}$ is dimensionless, so is $g^{\mu\nu}$. From the formula of the Christoffel symbols of the second kind it then follows that $\Gamma^\mu_{\nu\rho}$ and $\partial_\sigma \Gamma^\mu_{\nu\rho}$ have physical dimensions L^{-1} (SI: m^{-1}), respectively L^{-2} (SI: m^{-2}). a1: From the definition of $R^\rho_{\mu\sigma\nu}$ in terms of the Christoffel symbols of the second kind and their spatial derivatives it follows that L^{-2} (SI: m^{-2}) is also the physical dimension of $R^\rho_{\mu\sigma\nu}$. a2–a3: Since taking a trace does not affect physical dimension, $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$ and $R = g^{\mu\nu} R_{\mu\nu}$ also have dimension L^{-2} (SI: m^{-2}). a4: Inserting the dimensions of G and c it then follows that $T_{\mu\nu}$ has the dimension $\mathrm{ML^{-1}T^{-2}}$ (SI: $\mathrm{kg\,m^{-1}s^{-2}}$ (momentum flux), or $\mathrm{N\,m^{-2}}$ (pressure), or $\mathrm{J\,m^{-3}}$ (energy density)).

In order to understand the non-relativistic limit of (3–4) and relate it to the Newtonian model (1–2) we consider the weak field approximation, in which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,,$$

in which $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$ and all other $\eta_{\mu\nu} = 0$ in suitable coordinates ('Minkowski background metric') and $h_{\mu\nu}$ is a sufficiently small non-constant perturbation field.

(5) **b.** Show that $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$, in which $\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu$.

(5) **c.** Express the Christoffel symbols of the second kind in terms of the Minkowski background metric and the perturbation field, up to $\mathcal{O}(h)$ linear approximation in $h_{\mu\nu}$, $\partial_\rho h_{\mu\nu}$, and higher order derivatives.

We have

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(-\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} \right) \approx \frac{1}{2} \eta^{\rho\lambda} \left(-\partial_\lambda h_{\mu\nu} + \partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} \right) .$$

(10) **d.** ✍ Do the same for the Riemann tensor, the Ricci tensor and the Ricci scalar, and show that this

yields a linearized Einstein tensor

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \approx \frac{1}{2}\left(\partial_{\rho\mu}h^\rho_\nu - \partial_{\mu\nu}h^\rho_\rho - \partial_\rho\partial^\rho h_{\mu\nu} + \partial_\nu\partial^\rho h_{\mu\rho} - \eta_{\mu\nu}\partial^{\rho\sigma}h_{\rho\sigma} + \eta_{\mu\nu}\partial_\rho\partial^\rho h^\sigma_\sigma\right) .$$

[*Hint:* Avoid redundant computations by considering relevant $\mathcal{O}(h)$ terms only.]

Riemann tensor:

$$R^\rho_{\mu\sigma\nu} = \partial_\sigma\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\mu\sigma} \approx \frac{1}{2}\left(\partial_{\sigma\mu}h^\rho_\nu - \partial_{\mu\nu}h^\rho_\sigma - \partial_\sigma\partial^\rho h_{\mu\nu} + \partial_\nu\partial^\rho h_{\mu\sigma}\right) ,$$

in which index raising is carried out with the constant dual Minkowski background metric $\eta^{\mu\nu}$, i.e. $h^\mu_\nu \doteq \eta^{\mu\rho}h_{\rho\nu}$ and $\partial^\mu \doteq \eta^{\mu\nu}\partial_\nu$. Note that all Γ-symbols are of order $\mathcal{O}(h)$, thus in particular we may discard all $\mathcal{O}(hh)$ $\Gamma\Gamma$-terms in the linear approximation. Ricci tensor:

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu} \approx \frac{1}{2}\left(\partial_{\rho\mu}h^\rho_\nu - \partial_{\mu\nu}h^\rho_\rho - \partial_\rho\partial^\rho h_{\mu\nu} + \partial_\nu\partial^\rho h_{\mu\rho}\right) .$$

Ricci scalar:

$$R = g^{\mu\nu}R_{\mu\nu} \approx \partial^{\mu\nu}h_{\mu\nu} - \partial_\mu\partial^\mu h^\nu_\nu .$$

Putting things together, and using once more $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(h)$, we obtain for the linearized Einstein tensor

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \approx \frac{1}{2}\left(\partial_{\rho\mu}h^\rho_\nu - \partial_{\mu\nu}h^\rho_\rho - \partial_\rho\partial^\rho h_{\mu\nu} + \partial_\nu\partial^\rho h_{\mu\rho} - \eta_{\mu\nu}\partial^{\rho\sigma}h_{\rho\sigma} + \eta_{\mu\nu}\partial_\rho\partial^\rho h^\sigma_\sigma\right) .$$

We assume mass to be statically distributed over a compact region, i.e. $\rho = \rho(x, y, z)$ independent of t and zero if $x^2+y^2+z^2 > R^2$ for some $R>0$.

(5) **e.** Show that Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu}$, provides an *exact* solution of the Einstein equation (4) in vacuum (i.e. *all* $T_{\mu\nu}=0$ on the r.h.s.).

For this we need to verify that $g_{\mu\nu} = \eta_{\mu\nu}$ satisfies (4) when $T_{\mu\nu} = 0$, in which case the p.d.e. system reduces to $R_{\mu\nu} = 0$ (contract with $g^{\mu\nu}$ to see this). Since $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ with $R^\rho_{\mu\sigma\nu} = \partial_\sigma\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\mu\sigma}$, and since all Γ-symbols vanish identically in Minkowski coordinates due to constancy of $\eta_{\mu\nu}$, it follows that $R_{\mu\nu} = 0$.

For the stationary case we stipulate that $g_{\mu\nu}$ is time independent. Moreover, we postulate (without further motivation here) a perturbation field of the form $h_{\mu\nu} = \Psi\delta_{\mu\nu}$, in which Ψ is some scalar field and $\delta_{\mu\nu}$ the 4-dimensional Kronecker symbol.

(5) **f.** ✍ Show that the $(\mu,\nu)=(0,0)$ component of the linearized Einstein equation (4) reduces to Newton's equation (2), and establish a relation between Ψ and the Newtonian potential Φ.

Evaluating the $(\mu,\nu)=(0,0)$-component of the linearized Einstein tensor obtained in problem d we find

$$R_{00} - \frac{1}{2}\eta_{00}R \approx \frac{1}{2}\left(\partial_{\rho0}h^\rho_0 - \partial_{00}h^\rho_\rho - \partial_\rho\partial^\rho h_{00} + \partial_0\partial^\rho h_{0\rho} - \eta_{00}\partial^{\rho\sigma}h_{\rho\sigma} + \eta_{00}\partial_\rho\partial^\rho h^\sigma_\sigma\right) .$$

Inserting $\eta_{00} = \eta^{00} = -1$, $\eta_{ij} = \eta^{ij} = \delta_{ij}$, $\partial_0 = c^{-1}\partial_t$, $\partial^0 = -c^{-1}\partial_t$, $\partial^i = \partial_i$, and dropping all ∂_t-derivatives, yields

$$R_{00} - \frac{1}{2}\eta_{00}R \approx -\Delta\Psi .$$

Comparison with (2) and (4), using $T_{00} = \rho c^2$, viz.

$$\frac{2}{c^2}\Delta\Phi \overset{(2)}{=} \frac{2}{c^2}4\pi G\rho = \frac{8\pi G}{c^4}\rho c^2 \overset{(4)\&\mathrm{d}}{\approx} R_{00} - \frac{1}{2}\eta_{00}R \overset{\mathrm{f}}{=} -\Delta\Psi ,$$

allows us to conclude

$$\Phi = -\frac{1}{2}c^2\Psi + H\,,$$

in which H is an arbitrary harmonic function, i.e. $\Delta H = 0$. The latter can be adjusted to asymptotic conditions; we set $H = 0$. Using $g_{00} = -1 + h_{00}$, we may rewrite this as

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right)\,.$$

(The motivation for the ansatz $h_{\mu\nu} = \Psi\delta_{\mu\nu}$ requires a consistency check by verifying all other components of the Einstein equation.)

Light velocity c does not enter in the Newtonian theory. It is stipulated that Newton's theory emerges from Einstein's theory in the limit $v/c \to 0$ under the weak field assumption.

g. Before establishing the kinematic relationship between (1) and (3), consider the following questions:

(2) **g1.** Argue why $V^i \ll V^0$ in the Newtonian limit.
[*Hint:* Recall the relation between relativistic 4-velocity V^μ and classical 3-velocity v^i.]

$V^\mu = (V^0; V^i) = \gamma(v)(c; v^i)$, whence $V^i = \gamma(v)v^i \ll \gamma(v)c = V^0$. Note that $v^i \ll c$ amounts to $\gamma(v) \approx 1 \doteq \gamma(0)$.

(2) **g2.** Show that this implies that we may approximate (3) by $\dfrac{dV^\mu}{d\tau} + \Gamma^\mu_{00}\left(V^0\right)^2 = 0$.

All Γ-symbols are $\mathcal{O}(h)$ in the weak field approximation. Based on g1, Γ-terms with $\nu = \rho = 0$ in (3) will dwarf all other terms in the Newtonian limit $v/c \to 0$.

(2) **g3.** Use the weak field approximation to show that $\dfrac{dV^\mu}{d\tau} = \dfrac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}\left(V^0\right)^2$.

This follows by inserting the $\mathcal{O}(h)$-approximation of Γ^μ_{00} as derived in problem c in the expression under g2.

(2) **g4.** Argue why we may assume V^0 to be approximately constant if the gravitational field is static.

Taking $\mu = 0$ in the equation under g3 and using the static assumption $\partial_0 h_{00} = 0$ we find $\dfrac{dV^0}{d\tau} = 0$.

(2) **g5.** Argue why this explains Newton's 'absolute time' paradigm.

Recall that $V^\mu = dx^\mu/d\tau$, whence $V^0 = dx^0/d\tau = c\,dt/d\tau =$ constant. Since the parameter τ is observer independent, any observer will see the same steady rate of progression of time t up to an arbitrary offset ('absolute time'). Note that, by definition, the constant equals $c\gamma(0) = c$, cf. g1. Thus with a suitable offset to gauge clocks, observer time t will always coincide with observer independent proper time τ.

(5) **h.** Establish the relation between (1) and (3) by relating the classical scalar gravitational potential Φ to the relativistic metric tensor $g_{\mu\nu}$ in the Newtonian limit. Is it consistent with your result under f?
[*Hint:* Consider $\mu = i = 1, 2, 3$ in g3 in the Newtonian limit.]

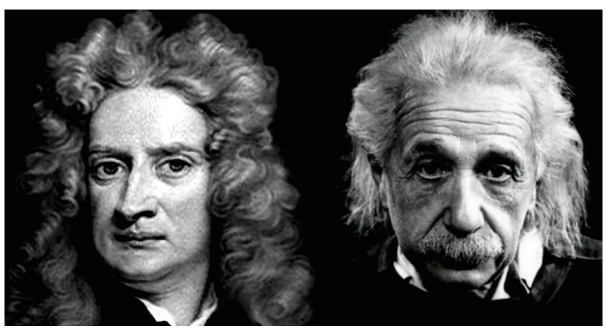
Take $t = \tau$ (recall g5) so that we may replace $d/d\tau \to d/dt$, $\gamma \to 1$, $V^0 \to c$ and $V^i \to v^i$ in the equation under g3. Then for $\mu = i = 1, 2, 3$ we obtain, using diagonality of the dual metric $\eta^{\mu\nu}$,

$$\frac{dv^i}{dt} = \frac{1}{2}c^2\eta^{ij}\partial_j h_{00}\,.$$

Comparison with (1) suggests that we take $\partial_j\Phi = -\frac{1}{2}c^2\partial_j h_{00}$. We may therefore take $\Phi = -\frac{1}{2}c^2 h_{00}$ (up to an integration constant), i.e.

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right)\,,$$

consistent with the result under f. Note that one is always free to add a constant to the potential in the kinematic equations, since only derivatives have physical significance.



NEWTON (1643–1727) AND EINSTEIN (1879–1955).

THE END