EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0. Date: Tuesday April 9, 2019. Time: 09h00–12h00. Place: Atlas 8.340.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. Credits are indicated in the margin.
- Items marked with ✍ may be relatively time consuming. It is safe to postpone them until the end.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No other material or equipment may be used.
- The Einstein summation convention is in effect throughout this exam.

(**25**) **1. TENSOR CALCULUS MISCELLANY.**

We consider a 2-dimensional real positive definite inner product space V with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. The metric tensor is given by $\mathbf{g} = g_{ij}\, \hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}^j$.

Let $A^{\text{cov}} = A_{ij}\, \hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}^j$ be a real symmetric covariant tensor relative to the dual basis $\{\hat{\mathbf{e}}^1, \hat{\mathbf{e}}^2\}$, and $A^{\text{mix}} = A^i_j\, \mathbf{e}_i \otimes \hat{\mathbf{e}}^j$ the associated mixed tensor with $A^i_j = g^{ik} A_{kj}$, with $g^{ik} g_{kj} = \delta^i_j$.

The 2×2 matrix $\mathbf{A}^{\text{cov}}$ is defined in terms of the real-valued holor A_{ij} of A^{cov}, viz.

$$\mathbf{A}^{\text{cov}} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \text{with } A_{21} = A_{12}.$$

The 2×2 matrix $\mathbf{A}^{\text{mix}}$ is defined in terms of the holor A^i_j of $A^{\text{mix}} = A^i_j\, \mathbf{e}_i \otimes \hat{\mathbf{e}}^j$, viz.

$$\mathbf{A}^{\text{mix}} = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix}.$$

(5) **a.** Is the matrix $\mathbf{A}^{\text{mix}}$ symmetric?

Below we consider the generic situation in which the matrix $\mathbf{A}^{\text{cov}}$ is nonsingular, so that $\det \mathbf{A}^{\text{cov}} \neq 0$.

(5) **b.** Does this imply that the matrix $\mathbf{A}^{\text{mix}}$ is nonsingular?

Definition. An eigenvector $v = v^i \mathbf{e}_i \in V$ of A^{mix} is a nonzero vector satisfying $\mathbf{A}^{\text{mix}}\,\mathbf{v} = \lambda \mathbf{v}$ for some eigenvalue $\lambda \in \mathbb{C}$, in which $\mathbf{v} \in \mathbb{C}^2$ is the column vector defined in terms of the holor v^i of $\mathbf{v}$, i.e.

$$\mathbf{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} .$$

Thus in the sense of classical matrix theory the column vector $\mathbf{v} \in \mathbb{C}^2$ is an eigenvector of the matrix $\mathbf{A}^{\text{mix}}$ with eigenvalue $\lambda \in \mathbb{C}$.

(5) **c.** Show that, if $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A}^{\text{mix}}$, then $\det(\mathbf{A}^{\text{mix}} - \lambda \mathbf{I}) = 0$, in which $\mathbf{I}$ is the 2×2 identity matrix.

Definition. The trace tr $\mathbf{A}^{\text{mix}}$ is the sum of the diagonal elements of $\mathbf{A}^{\text{mix}}$, i.e. tr $\mathbf{A}^{\text{mix}} = A^i_i$.

(5) **d.** Show that, in our 2-dimensional case and for general $\lambda \in \mathbb{C}$ (not necessarily an eigenvalue),

$$\det(\mathbf{A}^{\text{mix}} - \lambda \mathbf{I}) = \lambda^2 - \operatorname{tr} \mathbf{A}^{\text{mix}}\, \lambda + \det \mathbf{A}^{\text{mix}} .$$

[*Hint:* Recall $\delta^{k\ell}_{ij} = [ij][k\ell] = \delta^k_i \delta^\ell_j - \delta^k_j \delta^\ell_i$.]

Definition. We define the set S of scalar invariants associated with the tensor A^{mix} as follows:

$$S = \{ t_k \in \mathbb{R} \mid t_k \doteq \operatorname{tr}(\underbrace{\mathbf{A}^{\text{mix}} \ldots \mathbf{A}^{\text{mix}}}_{k\text{-th matrix power}}) = \delta^{\ell_{k+1}}_{\ell_1} A^{\ell_1}_{\ell_2} A^{\ell_2}_{\ell_3} \ldots A^{\ell_k}_{\ell_{k+1}} \}_{k \in \mathbb{N}} .$$

(5) **e.** Prove that each t_k is indeed invariant in the sense of being independent of the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of V.

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(25) 2. Area Two-Forms.

The wedge product of two covectors $\hat{\mathbf{v}}, \hat{\mathbf{w}} \in V^*$, i.e. $\hat{\mathbf{v}} \wedge \hat{\mathbf{w}}$, may be interpreted as an antisymmetric cotensor of rank 2, i.e. a bilinear map $\hat{\mathbf{v}} \wedge \hat{\mathbf{w}} : V \times V \to \mathbb{R} : (\mathbf{a}, \mathbf{b}) \mapsto (\hat{\mathbf{v}} \wedge \hat{\mathbf{w}})(\mathbf{a}, \mathbf{b})$, given by

$$(\hat{\mathbf{v}} \wedge \hat{\mathbf{w}})(\mathbf{a}, \mathbf{b}) = \det \begin{pmatrix} \langle \hat{\mathbf{v}}, \mathbf{a} \rangle & \langle \hat{\mathbf{v}}, \mathbf{b} \rangle \\ \langle \hat{\mathbf{w}}, \mathbf{a} \rangle & \langle \hat{\mathbf{w}}, \mathbf{b} \rangle \end{pmatrix} .$$

That is, $\hat{\mathbf{v}} \wedge \hat{\mathbf{w}} \in V^* \otimes_A V^* \doteq \bigwedge_2(V)$.

Similarly we may define the wedge product of two vectors, $\mathbf{a}, \mathbf{b} \in V$, i.e. $\mathbf{a} \wedge \mathbf{b}$, as an antisymmetric contratensor of rank 2, i.e. a bilinear map $\mathbf{a} \wedge \mathbf{b} : V^* \times V^* \to \mathbb{R} : (\hat{\mathbf{v}}, \hat{\mathbf{w}}) \mapsto (\mathbf{a} \wedge \mathbf{b})(\hat{\mathbf{v}}, \hat{\mathbf{w}})$, given by

$$(\mathbf{a} \wedge \mathbf{b})(\hat{\mathbf{v}}, \hat{\mathbf{w}}) = \det \begin{pmatrix} \langle \hat{\mathbf{v}}, \mathbf{a} \rangle & \langle \hat{\mathbf{w}}, \mathbf{a} \rangle \\ \langle \hat{\mathbf{v}}, \mathbf{b} \rangle & \langle \hat{\mathbf{w}}, \mathbf{b} \rangle \end{pmatrix} .$$

That is, $\mathbf{a} \wedge \mathbf{b} \in V \otimes_A V \doteq \bigwedge^2(V)$.

(10) **a.** Prove that $(\hat{\mathbf{v}} \wedge \hat{\mathbf{w}})(\mathbf{a}, \mathbf{b})$ actually depends only on $\mathbf{a} \wedge \mathbf{b}$, rather than independently on $\mathbf{a}$ and $\mathbf{b}$.

(10) **b.** Explain the meaning of the identification $V^* \otimes_A V^* = (V \otimes_A V)^*$.

We consider a linear transformation $L : V \times V \to V \times V : (\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{a}', \mathbf{b}') = L(\mathbf{a}, \mathbf{b})$, given by

$$\begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix} = \mathbf{L} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \quad \text{in which } \mathbf{L} \text{ is a } 2\times 2 \text{ matrix.}$$

(5) **c.** Under which conditions on $\mathbf{L}$ do we have $\mathbf{a} \wedge \mathbf{b} = \mathbf{a}' \wedge \mathbf{b}'$?

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(50) 3. NEWTONIAN LIMIT OF EINSTEIN GRAVITY.

Summation convention. In this problem we distinguish between two types of indices. Latin, or *spatial indices* $i, j, k, \ldots$ take values in the range $\{1, 2, 3\}$. Greek, or *spatiotemporal indices* $\mu, \nu, \rho, \ldots$ take values in the range $\{0, 1, 2, 3\}$. The summation convention for identical pairs of indices is in effect throughout this problem, with this tacit assumption on index range.

Gravity according to Newton. Newton regards spacetime as the product space of a flat Euclidean 3-space and 1-dimensional 'absolute time'. In 3-space inertial coordinates $(x^1, x^2, x^3) \doteq (x, y, z)$ exist such that the metric holor is $\eta_{ij} \doteq (\partial_i|\partial_j) = \delta_{ij}$ relative to the induced coordinate basis $\{\partial_i\}_{i=1,2,3}$. According to Newton, absolute time t progresses at a constant pace independently of any observer.

Kinematics. In Newtonian gravity one explains the response of matter to gravity in terms of a scalar gravitational potential field Φ entering Newton's second law of motion:

$$\frac{d^2x^i}{dt^2} = -\eta^{ij}\partial_j\Phi \,. \tag{1}$$

The classical 3-velocity has components $v^i \doteq dx^i/dt$ $(i = 1, 2, 3)$. The r.h.s. represents the classical gravitational force field $\mathrm{g}^i = -\eta^{ij}\partial_j\Phi$ experienced by a test particle.

Dynamics. In turn, Φ originates from the spatial distribution of matter via the Poisson equation:

$$\Delta\Phi = 4\pi G\rho \,. \tag{2}$$

Here $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}$ (in Cartesian coordinates) is the Laplacian operator, ρ denotes mass density and $G \approx 6.674\, 10^{-11}\, \mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is the universal gravitational constant.

Gravity according to Einstein. According to Einstein, on the other hand, the gravitational potential is a symmetric rank-2 covariant tensor field identified with the Lorentzian spacetime metric $g_{\mu\nu} = (\partial_\mu|\partial_\nu)$ with signature $(-, +, +, +)$ in 4-dimensional spacetime relative to a coordinate basis $\{\partial_\mu\}_{\mu=0,1,2,3}$ induced by arbitrary coordinates $(x^0, x^1, x^2, x^3) \doteq (ct, x, y, z)$, with $c \doteq 299\,792\,458\, \mathrm{ms}^{-1}$.

Kinematics. The relativistic counterpart of (1) takes the form of a geodesic equation dictated by the metric $g_{\mu\nu}$ via the Levi-Civita connection:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0 \,. \tag{3}$$

The observer independent affine parameter s is related to so-called 'proper time' τ via $s \doteq c\tau$. The vector with coefficients $V^\mu \doteq dx^\mu/d\tau$ is the (relativistic) 4-velocity, so we may equivalently write

$$\frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} V^\nu V^\rho = 0 \,. \tag{3}$$

Its relation to the classical 3-velocity $v^i = dx^i/dt$ ($i = 1, 2, 3$) is given by $(V^0; V^i) = \gamma(v)(c; v^i)$, with time dilation factor

$$\gamma(v) = \frac{dct}{ds} = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} \,,$$

depending on the particle's Euclidean 3-velocity magnitude v. Note that, despite its extra dimension, the 4-velocity V^μ has the same three degrees of freedom as the 3-velocity v^i.

Dynamics. The counterpart of (2) is the Einstein field equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{4}$$

in which $T_{\mu\nu}$ is the stress-energy tensor. The left hand side of (4) is known as the Einstein tensor. We henceforth consider a stationary matter-energy distribution, for which $T_{00} = \rho\, c^2$, in which ρ denotes mass density, and $T_{\mu\nu} = 0$ otherwise.

(5) **a.** Assuming the holor $g_{\mu\nu}$ of the metric $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$ to be dimensionless, determine the physical units of the tensors listed below. Write the relevant dimensions in terms of monomials in L (*length*), T (*time*), M (*mass*), or use corresponding SI units m (*meter*), s (*second*), kg (*kilogram*).

[*Example:* G has dimension $\mathrm{L}^3\mathrm{M}^{-1}\mathrm{T}^{-2}$ (SI: $\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$); $\|u\|^2 \doteq \mathbf{g}(u, u)$ has dimension L^2 (SI: m^2) if u is a spatiotemporal displacement vector, etc.]

a1. Riemann curvature tensor $R^\rho_{\mu\sigma\nu}$;

a2. Ricci curvature tensor $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$;

a3. Ricci curvature scalar $R = g^{\mu\nu} R_{\mu\nu}$;

a4. Stress-energy tensor $T_{\mu\nu}$.

In order to understand the non-relativistic limit of (3–4) and relate it to the Newtonian model (1–2) we consider the weak field approximation, in which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,,$$

in which $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$ and all other $\eta_{\mu\nu} = 0$ in suitable coordinates ('Minkowski background metric') and $h_{\mu\nu}$ is a sufficiently small non-constant perturbation field.

(5) **b.** Show that $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$, in which $\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu$.

(5) **c.** Express the Christoffel symbols of the second kind in terms of the Minkowski background metric and the perturbation field, up to $\mathcal{O}(h)$ linear approximation in $h_{\mu\nu}$, $\partial_\rho h_{\mu\nu}$, and higher order derivatives.

(10) **d.** ✍ Do the same for the Riemann tensor, the Ricci tensor and the Ricci scalar, and show that this

yields a linearized Einstein tensor

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \approx \frac{1}{2} \left(\partial_{\rho\mu} h^{\rho}_{\nu} - \partial_{\mu\nu} h^{\rho}_{\rho} - \partial_{\rho} \partial^{\rho} h_{\mu\nu} + \partial_{\nu} \partial^{\rho} h_{\mu\rho} - \eta_{\mu\nu} \partial^{\rho\sigma} h_{\rho\sigma} + \eta_{\mu\nu} \partial_{\rho} \partial^{\rho} h^{\sigma}_{\sigma} \right) .$$

[*Hint:* Avoid redundant computations by considering relevant $\mathcal{O}(h)$ terms only.]

We assume mass to be statically distributed over a compact region, i.e. $\rho = \rho(x, y, z)$ independent of t and zero if $x^2+y^2+z^2 > R^2$ for some $R>0$.

(5) **e.** Show that Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu}$, provides an *exact* solution of the Einstein equation (4) in vacuum (i.e. *all* $T_{\mu\nu}=0$ on the r.h.s.).

For the stationary case we stipulate that $g_{\mu\nu}$ is time independent. Moreover, we postulate (without further motivation here) a perturbation field of the form $h_{\mu\nu} = \Psi \delta_{\mu\nu}$, in which Ψ is some scalar field and $\delta_{\mu\nu}$ the 4-dimensional Kronecker symbol.

(5) **f.** ✍ Show that the $(\mu, \nu) = (0, 0)$ component of the linearized Einstein equation (4) reduces to Newton's equation (2), and establish a relation between Ψ and the Newtonian potential Φ.

Light velocity c does not enter in the Newtonian theory. It is stipulated that Newton's theory emerges from Einstein's theory in the limit $v/c \to 0$ under the weak field assumption.

g. Before establishing the kinematic relationship between (1) and (3), consider the following questions:

(2) **g1.** Argue why $V^i \ll V^0$ in the Newtonian limit.
[*Hint:* Recall the relation between relativistic 4-velocity V^μ and classical 3-velocity v^i.]

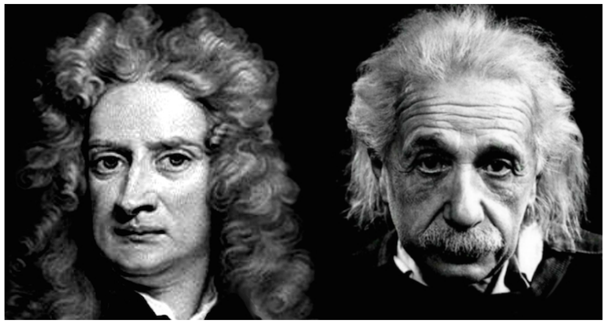
(2) **g2.** Show that this implies that we may approximate (3) by $\frac{dV^\mu}{d\tau} + \Gamma^\mu_{00} \left(V^0 \right)^2 = 0$.

(2) **g3.** Use the weak field approximation to show that $\frac{dV^\mu}{d\tau} = \frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00} \left(V^0 \right)^2$.

(2) **g4.** Argue why we may assume V^0 to be approximately constant if the gravitational field is static.

(2) **g5.** Argue why this explains Newton's 'absolute time' paradigm.

(5) **h.** Establish the relation between (1) and (3) by relating the classical scalar gravitational potential Φ to the relativistic metric tensor $g_{\mu\nu}$ in the Newtonian limit. Is it consistent with your result under f?
[*Hint:* Consider $\mu = i = 1, 2, 3$ in g3 in the Newtonian limit.]



NEWTON (1643–1727) AND EINSTEIN (1879–1955).

THE END