EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0. Date: Tuesday April 10, 2018. Time: 09h00–12h00. Place: VRT 4.15 B

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No other material or equipment may be used.

(35) 1. IDENTITIES INVOLVING DETERMINANTS.

Definition. A function $f \in C^1(\mathbb{R}^n)$ is called homogeneous of degree $\alpha \in \mathbb{R}$ if

$$f(\lambda x) = \lambda^\alpha f(x)\,,$$

for all $\lambda \in \mathbb{R}$.

(5) **a.** Show that such a function satisfies the p.d.e. $x^i \partial_i f(x) = \alpha f(x)$.

Differentiate left and right hand sides of the equation $f(\lambda x) = \lambda^\alpha f(x)$ with respect to λ and then put $\lambda = 1$.

The set of all $n \times n$ matrices A with $\mathbb{R}$-valued entries will be denoted by $\mathbb{M}_n$.

(5) **b.** Let $A \in \mathbb{M}_n$. Show that $\det(\lambda A) = \lambda^n \det A$.

We have $\det A = \epsilon^{i_1 \ldots i_n} A_{1i_1} \ldots A_{ni_n}$, so that $\det(\lambda A) = \epsilon^{i_1 \ldots i_n} (\lambda A_{1i_1}) \ldots (\lambda A_{ni_n}) = \lambda^n \epsilon^{i_1 \ldots i_n} A_{1i_1} \ldots A_{ni_n} = \lambda^n \det A$, in which we have defined $\epsilon^{i_1 \ldots i_n} = [i_1, \ldots i_n]$.

By cof$A \in \mathbb{M}_n$ and adj$A \in \mathbb{M}_n$ we denote the *cofactor*, respectively *adjugate matrix* of $A \in \mathbb{M}_n$:

$$(\mathrm{cof}A)^{ij} = \frac{\partial \det A}{\partial A_{ij}} \quad \text{resp.} \quad (\mathrm{adj}A)^{ij} = \frac{\partial \det A}{\partial A_{ji}}\,.$$

The *trace* tr $A \in \mathbb{R}$ of a matrix $A \in \mathbb{M}_n$ is the sum of its diagonal elements: $\mathrm{tr}\, A \stackrel{\text{def}}{=} A^i_i$.

Definition. Let $\boldsymbol{A} = A^i_j\, \boldsymbol{e}_i \otimes \hat{\boldsymbol{e}}^j$ be the mixed tensor corresponding to the matrix A relative to a fiducial basis. Relative to this basis we define $\mathrm{tr}\, \boldsymbol{A} \stackrel{\text{def}}{=} \mathrm{tr}\, A$.

(5) **c.** Show that tr A is invariant under basis transformations.

Suppose $\boldsymbol{f}_i = \Phi_i^j \boldsymbol{e}_j$, then $\hat{\boldsymbol{f}}^i = \Psi_j^i \hat{\boldsymbol{e}}^j$, in which $\Phi_k^i \Psi_j^k = \delta_j^i$. Suppose $\boldsymbol{A} = A_j^i \, \boldsymbol{e}_i \otimes \hat{\boldsymbol{e}}^j = \overline{A}_\ell^k \, \boldsymbol{f}_k \otimes \hat{\boldsymbol{f}}^\ell$. Substitution of the defining equations for the new basis (co-)vectors in terms of the original ones yiels $\boldsymbol{A} = \overline{A}_\ell^k \Phi_k^i \Psi_j^\ell \, \boldsymbol{e}_i \otimes \hat{\boldsymbol{e}}^j$, so that apparently $A_j^i = \overline{A}_\ell^k \Phi_k^i \Psi_j^\ell$ ('tensor transformation law'). Setting $i = j$ and applying summation convention we find tr $A = A_i^i = \overline{A}_\ell^k \Phi_k^i \Psi_i^\ell = \overline{A}_\ell^k \delta_k^\ell = \overline{A}_i^i = \text{tr}\, \overline{A}$.

(5) **d.** Prove: (i) $(\text{adj}A)A = \det A \, I_n$ and (ii) tr $((\text{adj}A)A) = n \det A$. I_n denotes the $n \times n$ identity matrix. [*Hint:* Use a & b.]

We have, using $\epsilon^{i_1 \cdots i_n} = [i_1, \ldots i_n]$,

$$\tilde{A}^{ki} A_{kj} = \frac{\partial \det A}{\partial A_{ki}} A_{kj} = \frac{\partial}{\partial A_{ki}} \epsilon^{i_1 \ldots i_n} A_{1 i_1} \ldots A_{n i_n} A_{kj} = \frac{1}{n!} \frac{\partial}{\partial A_{ki}} \left(\epsilon^{i_1 \ldots i_n} \epsilon^{j_1 \ldots j_n} A_{j_1 i_1} \ldots A_{j_n i_n} \right) A_{kj} = \star \, .$$

Applying the product rule, and indicating the omission of a factor $A_{j_\ell i_\ell}$ by $A_{\hat{\jmath}_\ell \hat{\imath}_\ell}$, we obtain

$$\star = \frac{1}{n!} \epsilon^{i_1 \ldots i_n} \epsilon^{j_1 \ldots j_n} \sum_{\ell=1}^n A_{j_1 i_1} \ldots A_{\hat{\jmath}_\ell \hat{\imath}_\ell} \ldots A_{j_n i_n} \delta_{j_\ell}^k \delta_{i_\ell}^i A_{kj} = \frac{1}{n!} \epsilon^{i_1 \ldots i_n} \epsilon^{j_1 \ldots j_n} \sum_{\ell=1}^n A_{j_1 i_1} \ldots A_{j_\ell j} \ldots A_{j_n i_n} \delta_{i_\ell}^i \, .$$

Upon inspection of the r.h.s. we conclude that the only nontrivial contribution in the product can come from the choice $j = i_\ell$, since only this index can differ from the $n-1$ remaining contraction indices $i_1, \ldots, \hat{\imath}_\ell, \ldots, i_n$ in view of the factor $\epsilon^{i_1 \cdots i_n}$ in front. But this implies that only $i = j$ will give a nontrivial result. Hence

$$\star = \frac{1}{n!} \epsilon^{i_1 \ldots i_n} \epsilon^{j_1 \ldots j_n} A_{j_1 i_1} \ldots A_{j_\ell i_\ell} \ldots A_{j_n i_n} \delta_j^i = \delta_j^i \det A \, .$$

Taking the trace immediately yields tr $((\text{adj}A)A) = \tilde{A}^{ki} A_{ki} = \delta_i^i \det A = n \det A$.

Assumption. In the remainder of this problem we consider matrices $A \in \mathbb{M}_n$ that are *positive-definite* and *symmetric*. Recall that such matrices can be diagonalized by a suitable rotation, say

$$\Delta \stackrel{\text{def}}{=} R^{\text{T}} A R \stackrel{\text{def}}{=} \text{diag}\,(\lambda_1, \ldots, \lambda_n) \, ,$$

in which $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ are the eigenvalues of A.

Definition. The anisotropic Gaussian function $\phi_A \in C^\infty(\mathbb{R}^n)$ is defined as

$$\phi_A(x) = c_A e^{-x^i A_{ij} x^j} \, ,$$

in which $A \in \mathbb{M}_n$ is a positive-definite symmetric matrix with entries A_{ij} and $c_A > 0$ some A-dependent normalization constant. Below you may use the following standard integral:

$$\int_{\mathbb{R}} e^{-z^2} \, dz = \sqrt{\pi} \, .$$

(5) **e.** Assume: $\displaystyle\int_{\mathbb{R}^n} \phi_A(x) \, dx = 1$. Find c_A in terms of A. [*Hint:* Substitute $x = Ry$ for a suitably chosen rotation matrix R.]

Following the hint we obtain, in matrix notation, using $\det R = 1$, and choosing R such that $\Delta \stackrel{\text{def}}{=} R^{\text{T}} A R \stackrel{\text{def}}{=} \text{diag}\,(\lambda_1, \ldots, \lambda_n)$ is diagonal,

$$\int_{\mathbb{R}^n} e^{-x \cdot Ax} \, dx = \int_{\mathbb{R}^n} e^{-y \cdot R^{\text{T}} A R y} \, dy = \int_{\mathbb{R}^n} e^{-y \cdot \Delta y} \, dy = \frac{1}{\sqrt{\lambda_1 \ldots \lambda_n}} \int_{\mathbb{R}^n} e^{-\|z\|^2} \, dz = \frac{1}{\sqrt{\det A}} \left(\int_{\mathbb{R}} e^{-z^2} \, dz \right)^n = \frac{\sqrt{\pi}^n}{\sqrt{\det A}} \, .$$

We thus have $c_A = \dfrac{\sqrt{\det A}}{\sqrt{\pi}^n}$.

Definition. For any analytical function $f \in C^\omega(\mathbb{R})$, with Taylor expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$ say, we define a matrix counterpart carrying the same name, $f \in C^\omega(\mathbb{M}_n)$, as follows:

$$f(A) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} a_k A^k \,.$$

($2\frac{1}{2}$) **f1.** Prove that, if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A, then $f(\lambda_1), \ldots, f(\lambda_n)$ are eigenvalues of $f(A)$.

Note that if v is an eigenvector of A with eigenvalue λ, then it is also an eigenvector of A^k, with eigenvalue λ^k: $A^k v = \lambda^k v$. As a result, $f(A)v = \sum_{k=0}^{\infty} a_k A^k v = \sum_{k=0}^{\infty} a_k \lambda^k v = f(\lambda)v$.

($2\frac{1}{2}$) **f2.** Prove that $f(R^{\mathrm{T}}AR) = R^{\mathrm{T}} f(A) R$.

The basic observation is that in the k-fold product $(R^{\mathrm{T}}AR)^k = R^{\mathrm{T}}AR\, R^{\mathrm{T}}AR \ldots R^{\mathrm{T}}AR$ products of type $R\,R^{\mathrm{T}} = I$ may be discarded. As a result, $(R^{\mathrm{T}}AR)^k = R^{\mathrm{T}}A^k R$, whence $f(R^{\mathrm{T}}AR) = \sum_{k=0}^{\infty} a_k (R^{\mathrm{T}}AR)^k = \sum_{k=0}^{\infty} a_k R^{\mathrm{T}} A^k R = R^{\mathrm{T}}(\sum_{k=0}^{\infty} a_k A^k) R = R^{\mathrm{T}} f(A) R$.

(5) **f3.** Show that $\det e^A = e^{\operatorname{tr} A}$.
[*Hint:* Start from $\det e^A = \det(R^{\mathrm{T}} e^A R)$.]

There exists a rotation matrix such that $R^{\mathrm{T}}AR = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \stackrel{\text{def}}{=} \Delta$. We have $\det e^A = \det(R^{\mathrm{T}} e^A R) \stackrel{\text{f2}}{=} \det e^{R^{\mathrm{T}}AR} = \det e^{\Delta} = \det e^{\operatorname{diag}(\lambda_1, \ldots, \lambda_n)} = \det \operatorname{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n}) = e^{\lambda_1} \ldots e^{\lambda_n} = e^{\lambda_1 + \ldots + \lambda_n} = e^{\operatorname{tr} A}$.

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(30) 2. Scalar Fields and Gauge Fields in Electro-Magnetism.

In particle physics one considers $\mathbb{C}$-valued field quantities associated with manifestations of elementary particles. Perhaps the simplest case is that of a scalar field $\psi : \mathbb{R}^n \to \mathbb{C} : x \mapsto \psi(x)$ for a scalar boson.

Physical interactions are formulated in terms of so-called Lagrangians. For a scalar boson we stipulate a Lagrangian of the form

$$\mathscr{L}_{\text{boson}}(\psi, \partial\psi) = \partial^\mu \psi^* \partial_\mu \psi - m^2 \psi^* \psi \,,$$

in which $\partial^\mu = g^{\mu\nu}\partial_\nu$, with $g^{\mu\nu}$ the holor of the dual of a Lorentzian type metric tensor $\mathbf{G} = g_{\mu\nu} dx^\mu \otimes dx^\nu$, and in which $z^* \in \mathbb{C}$ denotes the complex conjugate of $z \in \mathbb{C}$.

(5) **a1.** Show that, if $x = x(\overline{x})$ denotes a coordinate transformation, then $\overline{\partial}_\mu = \dfrac{\partial x^\nu}{\partial \overline{x}^\mu} \partial_\nu$ and $\overline{\partial}^\mu = \dfrac{\partial \overline{x}^\mu}{\partial x^\nu} \partial^\nu$.

By the chain rule for partial derivatives we have $\overline{\partial}_\mu = \frac{\partial x^\nu}{\partial \overline{x}^\mu} \partial_\nu$. To raise the index we need to express $\overline{g}^{\mu\nu}$ in terms of $g^{\rho\sigma}$ and Jacobian matrices. To this end consider $\overline{g}^{\mu\nu} \overline{\partial}_\mu \otimes \overline{\partial}_\nu = g^{\rho\sigma} \partial_\rho \otimes \partial_\sigma$ and apply the chain rule to the right hand side so as to obtain $\overline{g}^{\mu\nu} \overline{\partial}_\mu \otimes \overline{\partial}_\nu = \frac{\partial \overline{x}^\mu}{\partial x^\rho} \frac{\partial \overline{x}^\nu}{\partial x^\sigma} g^{\rho\sigma} \overline{\partial}_\mu \otimes \overline{\partial}_\nu$, from which we read off $\overline{g}^{\mu\nu} = \frac{\partial \overline{x}^\mu}{\partial x^\rho} \frac{\partial \overline{x}^\nu}{\partial x^\sigma} g^{\rho\sigma}$. As a result, $\overline{\partial}^\mu = \overline{g}^{\mu\nu} \overline{\partial}_\nu = \frac{\partial \overline{x}^\mu}{\partial x^\rho} \frac{\partial \overline{x}^\nu}{\partial x^\sigma} g^{\rho\sigma} \frac{\partial x^\lambda}{\partial \overline{x}^\nu} \partial_\lambda = \frac{\partial \overline{x}^\mu}{\partial x^\nu} \partial^\nu$. In the last step we have used, besides trivial dummy switches, $\frac{\partial \overline{x}^\nu}{\partial x^\sigma} \frac{\partial x^\lambda}{\partial \overline{x}^\nu} = \delta^\lambda_\sigma$ and the definition of $\partial^\nu = g^{\nu\lambda} \partial_\lambda$.

Definition (Scalar Field). Let $u : \mathbb{R}^n \to \mathbb{C} : x \mapsto u(x)$ be a scalar field. If $x = x(\overline{x})$ is a coordinate transformation, we set $\overline{u}(\overline{x}) \stackrel{\text{def}}{=} u(x)$.

(5) **a2.** Show that $\mathscr{L}_{\text{boson}}$ is invariant under coordinate transformations, i.e. $\mathscr{L}_{\text{boson}} = \overline{\mathscr{L}}_{\text{boson}}$.

Note that the invariance requirement $\mathscr{L}_{\text{boson}} = \overline{\mathscr{L}}_{\text{boson}}$ is equivalent to $\mathscr{L}_{\text{boson}}(\overline{\psi}, \overline{\partial\psi}) = \mathscr{L}_{\text{boson}}(\psi, \partial\psi)$ (in which the left hand side is considered as a function of $\overline{x}$ and the right hand side as a function of x). To verify this, consider $\mathscr{L}_{\text{boson}}(\overline{\psi}, \overline{\partial\psi}) = \overline{\partial}^\mu \overline{\psi}^* \overline{\partial}_\mu \overline{\psi} - m^2 \overline{\psi}^* \overline{\psi} \stackrel{\text{a1}}{=} \frac{\partial \overline{x}^\mu}{\partial x^\nu} \partial^\nu \psi^* \frac{\partial x^\rho}{\partial \overline{x}^\mu} \partial_\rho \psi - m^2 \psi^* \psi = \partial^\mu \psi^* \partial_\mu \psi - m^2 \psi^* \psi$. In the second last step we have used a1 and the fact that ψ is a scalar field (i.e. $\overline{\psi}(\overline{x}) = \psi(x)$), and in the last step we have used, besides dummy switches, the identity $\frac{\partial \overline{x}^\mu}{\partial x^\nu} \frac{\partial x^\rho}{\partial \overline{x}^\mu} = \delta^\rho_\nu$.

(5) **b1.** Show that $\mathscr{L}_{\text{boson}}$ is invariant under *global* phase shifts of the type $\widetilde{\psi}(x) = e^{i\alpha}\psi(x)$, with $\alpha \in \mathbb{R}$ constant.

For the zeroth order term in the Lagrangian we have $\widetilde{\psi}^* \widetilde{\psi} = e^{-i\alpha} \psi^* e^{i\alpha} \psi = \psi^* \psi$. Since α does not depend on x we also have, for the first order term, $\partial^\mu \widetilde{\psi}^* \partial_\mu \widetilde{\psi} = e^{-i\alpha} \partial^\mu \psi^* e^{i\alpha} \partial_\mu \psi = \partial^\mu \psi^* \partial_\mu \psi$. Thus $\mathscr{L}_{\text{boson}}(\psi, \partial\psi) = \mathscr{L}_{\text{boson}}(\widetilde{\psi}, \partial\widetilde{\psi})$.

(5) **b2.** Show that $\mathscr{L}_{\text{boson}}$ is *not* invariant under *local* phase shifts, i.e. $\widetilde{\psi}(x) = e^{i\alpha(x)}\psi(x)$, in which now $\alpha \in C^\infty(\mathbb{R}^n)$ represents a smooth real-valued scalar function.

Now the first order term in the Lagrangian is no longer invariant. We have extra terms due to the x-dependence of the phase $\alpha(x)$, viz. $\delta\mathscr{L}_{\text{boson}}(\psi, \partial\psi) \stackrel{\text{def}}{=} \mathscr{L}_{\text{boson}}(\widetilde{\psi}, \partial\widetilde{\psi}) - \mathscr{L}_{\text{boson}}(\psi, \partial\psi) = i\partial^\mu\alpha(\partial_\mu\psi^*\psi - \psi^*\partial_\mu\psi) + \partial^\mu\alpha\partial_\mu\alpha\psi^*\psi$.

In order to render the theory invariant under local phase shifts, a remedy has been proposed that requires the introduction of a so-called *gauge field*, as follows. Instead of ordinary partial derivatives $\partial_\mu\psi$, consider covariant derivatives of the form $D^{\text{A}}_\mu\psi = \partial_\mu\psi + iqA_\mu\psi$, in which q is a "coupling constant" (a.k.a. "charge") and in which $A_\mu : \mathbb{R}^n \to \mathbb{C} : x \mapsto A_\mu(x)$ represents the (holor of the) gauge field.

To enforce the desired invariance we require $\widetilde{D}^{\text{A}}_\mu \widetilde{\psi}(x) = e^{i\alpha(x)} D^{\text{A}}_\mu \psi(x)$ given the local transformation $\widetilde{\psi}(x) = e^{i\alpha(x)}\psi(x)$ considered in b2. Here $\widetilde{D}^{\text{A}}_\mu = \partial_\mu + iq\widetilde{A}_\mu$ denotes the appropriately transformed covariant derivative.

(5) **c.** Show that local phase transformations $\widetilde{\psi} = e^{i\alpha}\psi$ induce gauge field transformations $\widetilde{A}_\mu = A_\mu - \frac{1}{q}\partial_\mu\alpha$.

In order to guarantee local phase invariance by a formal replacement of $\partial_\mu \to D^{\text{A}}_\mu$ in the Lagrangian, we require $\widetilde{D}^{\text{A}}_\mu \widetilde{\psi} = e^{i\alpha} D^{\text{A}}_\mu \psi$. The right hand side equals $e^{i\alpha}(\partial_\mu + iqA_\mu)\psi$. The left hand side equals $e^{i\alpha}(\partial_\mu + i\partial_\mu\alpha + iq\widetilde{A}_\mu)\psi$. Enforcing equality we find for the gauge field $\widetilde{A}_\mu = A_\mu - \frac{1}{q}\partial_\mu\alpha$.

If the gauge field has any dynamics of its own, it must be incorporated into the Lagrangian in a gauge invariant way. It is stipulated that this is accomplished by adding a term

$$\mathscr{L}_{\text{E.M.}}(A) = -\frac{1}{4} F_{\mu\nu}(A) F^{\mu\nu}(A)$$

to the foregoing Lagrangian (besides the replacement $\partial_\mu \to D^{\text{A}}_\mu$), in which

$$F_{\mu\nu}(A) \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

(5) **d.** Show that $F_{\mu\nu}(A)$ (and thus $\mathscr{L}_{\text{E.M.}}(A)$) is invariant under gauge field transformations, recall c.

We have $\widetilde{F}_{\mu\nu} = \partial_\mu \widetilde{A}_\nu - \partial_\nu \widetilde{A}_\mu \stackrel{\text{c}}{=} \partial_\mu(A_\nu - \frac{1}{q}\partial_\nu\alpha) - \partial_\nu(A_\mu - \frac{1}{q}\partial_\mu\alpha) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$. Alternatively, note that we may replace ∂ in the definition of $F_{\mu\nu}$ by D^{A}: $F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu = (\partial_\mu + iqA_\mu)A_\nu - (\partial_\nu + iqA_\nu)A_\mu = D^{\text{A}}_\mu A_\nu - D^{\text{A}}_\nu A_\mu$.

☞ All in all, gauge invariance of the bosonic Lagrangian requires a gauge field in the form of the 'electro-magnetic' 4-vector potential A_μ, and the full gauge invariant Lagrangian becomes

$$\mathscr{L}(\psi, \partial\psi, A) = \mathscr{L}_{\text{boson}}(\psi, D^{\text{A}}\psi) + \mathscr{L}_{\text{E.M.}}(A) = D_{\text{A}}^{\mu}\psi^* D_{\mu}^{\text{A}}\psi - m^2\psi^*\psi - \frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A)\,.$$

A is the so-called electromagnetic 4-vector potential, a relativistic vector function combining an electric scalar potential ϕ and a magnetic 3-vector potential $\boldsymbol{A}$ into a single four-vector $A = (\phi/c, \boldsymbol{A})$. The EM-fields $(\boldsymbol{E}, \boldsymbol{B})$ follow from these potentials by

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \quad \text{respectively} \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}\,.$$

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(35) 3. LEVI-CIVITA CONNECTION & COVARIANT DERIVATIVE[1].

In this problem we consider a vector field $v(x) = v^i(x)\partial_i$ on an n-dimensional Riemannian manifold $\mathbb{M}$, locally decomposed relative to a coordinate basis $\{\partial_i = \partial/\partial x^i\}_{i=1,\ldots,n}$ at each basepoint $x \in \mathbb{M}$. The Riemannian metric tensor field is given by $G(x) = g_{ij}(x)dx^i \otimes dx^j$, with $\langle dx^i, \partial_j\rangle = \delta^i_j$.

We are interested in the rate of change of the vector field, $dv(x(t))/dt$, along a parametrized curve $x : \mathbb{R} \to \mathbb{M} : t \mapsto x(t)$. The local tangent at $x(t) \in \mathbb{M}$ is denoted by $\dot{x}(t) = \dot{x}^i(t)\partial_i$, with $\dot{} \equiv d/dt$.

A coordinate transformation $x = x(\overline{x})$ induces a local basis transformation, with transformed basis $\{\overline{\partial}_i = \partial/\partial\overline{x}^i\}_{i=1,\ldots,n}$ and Jacobian matrices

$$A^i_j(\overline{x}) = \frac{\partial x^i(\overline{x})}{\partial \overline{x}^j} \qquad \text{and} \qquad B^i_j(x) = \frac{\partial \overline{x}^i(x)}{\partial x^j}$$

Treat A^i_j and B^i_j implicitly as functions of $\overline{x}$, resp. x (as indicated explicitly in their definitions above).

($2\frac{1}{2}$) **a1.** Show that $v^i = A^i_j\overline{v}^j$ ('vector transformation').

We have $v^i\partial_i = \overline{v}^j\overline{\partial}_j \overset{\text{c.r.}}{=} \overline{v}^j\frac{\partial x^i}{\partial\overline{x}^j}\partial_i = \overline{v}^jA^i_j\partial_i$, whence $v^i = A^i_j\overline{v}^j$. The step marked by 'c.r.' relies on the chain rule.

($2\frac{1}{2}$) **a2.** Show that $\overline{g}_{ij} = A^k_iA^\ell_jg_{k\ell}$.

We have $\overline{g}_{ij}d\overline{x}^i \otimes d\overline{x}^j = g_{k\ell}dx^k \otimes dx^\ell \overset{\text{c.r.}}{=} g_{k\ell}\frac{\partial x^k}{\partial\overline{x}^i}\frac{\partial x^\ell}{\partial\overline{x}^j}d\overline{x}^i \otimes d\overline{x}^j = A^k_iA^\ell_jg_{k\ell}d\overline{x}^i \otimes d\overline{x}^j$, from which we deduce that $\overline{g}_{ij} = A^k_iA^\ell_jg_{k\ell}$.

($2\frac{1}{2}$) **a3.** Show that $\overline{\partial}_iA^k_j = \overline{\partial}_jA^k_i$.

This readily follows from the observation that $A^k_j = \overline{\partial}_jx^k$ and commutativity of partial differentiation.

($2\frac{1}{2}$) **a4.** Show that $\overline{g}_{k\ell}B^\ell_m = g_{m\ell}A^\ell_k$.

This follows from a2: $\overline{g}_{k\ell}B^\ell_m \overset{\text{a2}}{=} A^n_kA^p_\ell g_{np}B^\ell_m = g_{nm}A^n_k = g_{m\ell}A^\ell_k$. The last two steps rely on $A^p_\ell B^\ell_m = \delta^p_m$, respectively symmetry of the Gram matrix and dummy switches.

Below we shall write v^i and $\overline{v}^i$ as shorthands for $v^i(x(t))$ and $\overline{v}^i(\overline{x}(t))$.

[1] Adapted from: H. Rund, The Differential Geometry of Finsler Spaces, Springer-Verlag, 1959, Chapter III, § 2.

$(2\frac{1}{2})$ **b.** Show that the acceleration components $\dot{v}^i = dv^i/dt$ do *not* obey the vector transformation law.

Using $v^i = A^i_j \overline{v}^j$ we find, upon taking d/dt on both sides, $dv^i/dt = dA^i_j/dt\,\overline{v}^j + A^i_j\,d\overline{v}^j/dt = \overline{\partial}_k A^i_j\,\dot{\overline{x}}^k\,\overline{v}^j + A^i_j\,d\overline{v}^j/dt$. Only the last term on the r.h.s. agrees with the vector transformation law. Note that, in the anomalous term, $\overline{\partial}_k A^i_j = \overline{\partial}_{kj} x^i$ is symmetric w.r.t. $j \leftrightarrow k$.

$(2\frac{1}{2})$ **c.** Establish the relation between $\overline{\partial}_k \overline{g}_{ij}$ and $\partial_k g_{ij}$, likewise accounting for the role of the Jacobian A^i_j (and its derivatives). Do the partial derivatives $\partial_k g_{ij}$ constitute a tensor holor?

We have, using a2, $\overline{\partial}_k \overline{g}_{ij} = \overline{\partial}_k (A^\ell_i A^m_j g_{\ell m})$. The Jacobian is expressed as a function of $\overline{x}$, whereas the metric holor on the r.h.s. is most naturally seen as a function of x. Applying product and chain rules we therefore obtain, using $\overline{\partial}_k = A^n_k \partial_n$ whenever appropriate,

$$\overline{\partial}_k \overline{g}_{ij} = (A^m_j \overline{\partial}_k A^\ell_i + A^\ell_i \overline{\partial}_k A^m_j) g_{\ell m} + A^\ell_i A^m_j A^n_k \partial_n g_{\ell m} \,.$$

Only the last term should be present if $\partial_k g_{ij}$ were to form a tensor holor.

We define the Christoffel symbols of the first kind as follows: $\gamma_{ijk} \stackrel{\text{def}}{=} \dfrac{1}{2}\left(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik}\right)$.

(5) **d.** Prove: $\overline{\gamma}_{ijk} = A^\ell_i A^m_j A^n_k \gamma_{\ell m n} + B^\ell_m \overline{\partial}_k A^m_i \overline{g}_{j\ell}$.

Using c we have, by cyclic changes of the free indices,

$$\begin{aligned}
\overline{\partial}_k \overline{g}_{ij} &= (A^m_j \overline{\partial}_k A^\ell_i + A^\ell_i \overline{\partial}_k A^m_j) g_{\ell m} + A^\ell_i A^m_j A^n_k \partial_n g_{\ell m} \\
\overline{\partial}_i \overline{g}_{jk} &= (A^m_k \overline{\partial}_i A^\ell_j + A^\ell_j \overline{\partial}_i A^m_k) g_{\ell m} + A^\ell_j A^m_k A^n_i \partial_n g_{\ell m} \\
\overline{\partial}_j \overline{g}_{ki} &= (A^m_i \overline{\partial}_j A^\ell_k + A^\ell_k \overline{\partial}_j A^m_i) g_{\ell m} + A^\ell_k A^m_i A^n_j \partial_n g_{\ell m}
\end{aligned}$$

Subtracting the first from the sum of the second and third yields

$$\begin{aligned}
&\overline{\partial}_i \overline{g}_{jk} + \overline{\partial}_j \overline{g}_{ki} - \overline{\partial}_k \overline{g}_{ij} = \\
&\quad \left(A^\ell_j A^m_k A^n_i + A^\ell_k A^m_i A^n_j - A^\ell_i A^m_j A^n_k\right) \partial_n g_{\ell m} + \left(A^m_k \overline{\partial}_i A^\ell_j + A^\ell_j \overline{\partial}_i A^m_k + A^m_i \overline{\partial}_j A^\ell_k + A^\ell_k \overline{\partial}_j A^m_i - A^m_j \overline{\partial}_k A^\ell_i - A^\ell_i \overline{\partial}_k A^m_j\right) g_{\ell m}
\end{aligned}$$

Working out parentheses nine terms arise on the r.h.s. By virtue of $\overline{\partial}_i A^k_j = \overline{\partial}_j A^k_i$ (recall a3), two of these are identical (viz. 4th and 7th), while four others cancel pairwise (viz. 5th against 8th and 6th against 9th). Using the definition of the Christoffel symbols of the first kind we end up with

$$\overline{\gamma}_{ikj} = A^\ell_i A^m_j A^n_k \gamma_{\ell n m} + A^\ell_k \overline{\partial}_j A^m_i g_{\ell m}$$

Finally, using $g_{\ell m} A^\ell_k = \overline{g}_{k\ell} B^\ell_m$ (recall a4), we may rewrite this as

$$\overline{\gamma}_{ikj} = A^\ell_i A^m_j A^n_k \gamma_{\ell n m} + B^\ell_m \overline{\partial}_j A^m_i \overline{g}_{k\ell}$$

This is the desired result (after interchange of j and k).

(5) **e1.** Show that $\dot{v}^i = A^i_j \dot{\overline{v}}^j + \overline{\partial}_k A^i_m \dot{\overline{x}}^k \overline{v}^m$.

This follows by differentiating a1, taking into account product and chain rules.

(5) **e2.** Use d to show that $\overline{\partial}_k A^i_m \dot{\overline{x}}^k = A^i_\ell \overline{\gamma}^\ell_{mk} \dot{\overline{x}}^k - A^\ell_m \gamma^i_{\ell k} \dot{x}^k$, in which $\gamma^k_{ij} \stackrel{\text{def}}{=} g^{k\ell} \gamma_{i\ell j}$.
[*Hint:* Contract the identity in d with $A^q_p \overline{g}^{jp} \dot{\overline{x}}^k$.]

Follow the hint. Various identities and trivial dummy index replacements are needed to obtain the desired form. For instance, $A^i_\ell B^\ell_j = \delta^i_j$, $g^{mn} = A^n_p A^m_j \overline{g}^{jp}$, whence $A^m_j \overline{g}^{jp} = B^p_n g^{mn}$, etc. Slick use of such identities will reveal the result by isolating the term $\overline{\partial}_k A^q_i \dot{\overline{x}}^k$.

(5) **f.** Use e1 and e2 to show that $\dfrac{Dv^i}{dt} \stackrel{\text{def}}{=} \dfrac{dv^i}{dt} + \gamma^i_{jk} v^j \dot{x}^k$ transforms as a vector.

Substitution of e2 into e1, and using $\overline{v}^m = B^m_n v^n$ to eliminate the overlined holor in one of the terms (cf. a1), allows us to place original and transformed holors on either side of the equation, immediately yielding the tensor transformation law: $\dot{v}^i + \gamma^i_{\ell k} v^\ell \dot{x}^k = A^i_j \left(\dot{\overline{v}}^j + \overline{\gamma}^j_{\ell k} \overline{v}^\ell \dot{\overline{x}}^k\right)$.

THE END