EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0. Date: Tuesday April 10, 2018. Time: 09h00–12h00. Place: VRT 4.15 B

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No other material or equipment may be used.

(**35**) **1. IDENTITIES INVOLVING DETERMINANTS.**

Definition. A function $f \in C^1(\mathbb{R}^n)$ is called homogeneous of degree $\alpha \in \mathbb{R}$ if

$$f(\lambda x) = \lambda^\alpha f(x)\,,$$

for all $\lambda \in \mathbb{R}$.

(5) **a.** Show that such a function satisfies the p.d.e. $x^i \partial_i f(x) = \alpha f(x)$.

The set of all $n \times n$ matrices A with $\mathbb{R}$-valued entries will be denoted by $\mathbb{M}_n$.

(5) **b.** Let $A \in \mathbb{M}_n$. Show that $\det(\lambda A) = \lambda^n \det A$.

By cof$A \in \mathbb{M}_n$ and adj$A \in \mathbb{M}_n$ we denote the *cofactor*, respectively *adjugate matrix* of $A \in \mathbb{M}_n$:

$$(\mathrm{cof}A)^{ij} = \frac{\partial \det A}{\partial A_{ij}} \quad \text{resp.} \quad (\mathrm{adj}A)^{ij} = \frac{\partial \det A}{\partial A_{ji}}\,.$$

The *trace* tr $A \in \mathbb{R}$ of a matrix $A \in \mathbb{M}_n$ is the sum of its diagonal elements: tr $A \stackrel{\text{def}}{=} A^i_i$.

Definition. Let $\boldsymbol{A} = A^i_j\, \boldsymbol{e}_i \otimes \hat{\boldsymbol{e}}^j$ be the mixed tensor corresponding to the matrix A relative to a fiducial basis. Relative to this basis we define tr $\boldsymbol{A} \stackrel{\text{def}}{=}$ tr A.

(5) **c.** Show that tr $\boldsymbol{A}$ is invariant under basis transformations.

(5) **d.** Prove: (i) $(\mathrm{adj}A)A = \det A\, I_n$ and (ii) tr $((\mathrm{adj}A)A) = n \det A$. I_n denotes the $n{\times}n$ identity matrix. [*Hint:* Use a & b.]

Assumption. In the remainder of this problem we consider matrices $A \in \mathbb{M}_n$ that are *positive-definite* and *symmetric*. Recall that such matrices can be diagonalized by a suitable rotation, say

$$\Delta \stackrel{\text{def}}{=} R^{\mathrm{T}} A R \stackrel{\text{def}}{=} \operatorname{diag}(\lambda_1, \ldots, \lambda_n) ,$$

in which $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ are the eigenvalues of A.

Definition. The anisotropic Gaussian function $\phi_A \in C^\infty(\mathbb{R}^n)$ is defined as

$$\phi_A(x) = c_A e^{-x^i A_{ij} x^j} ,$$

in which $A \in \mathbb{M}_n$ is a positive-definite symmetric matrix with entries A_{ij} and $c_A > 0$ some A-dependent normalization constant. Below you may use the following standard integral:

$$\int_{\mathbb{R}} e^{-z^2} \, dz = \sqrt{\pi} .$$

(5) **e.** Assume: $\displaystyle\int_{\mathbb{R}^n} \phi_A(x) \, dx = 1$. Find c_A in terms of A.
[*Hint:* Substitute $x = Ry$ for a suitably chosen rotation matrix R.]

Definition. For any analytical function $f \in C^\omega(\mathbb{R})$, with Taylor expansion $f(x) = \displaystyle\sum_{k=0}^{\infty} a_k x^k$ say, we define a matrix counterpart carrying the same name, $f \in C^\omega(\mathbb{M}_n)$, as follows:

$$f(A) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} a_k A^k .$$

($2\frac{1}{2}$) **f1.** Prove that, if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A, then $f(\lambda_1), \ldots, f(\lambda_n)$ are eigenvalues of $f(A)$.

($2\frac{1}{2}$) **f2.** Prove that $f(R^{\mathrm{T}} A R) = R^{\mathrm{T}} f(A) R$.

(5) **f3.** Show that $\det e^A = e^{\operatorname{tr} A}$.
[*Hint:* Start from $\det e^A = \det(R^{\mathrm{T}} e^A R)$.]

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(**30**) 2. Scalar Fields and Gauge Fields in Electro-Magnetism.

In particle physics one considers $\mathbb{C}$-valued field quantities associated with manifestations of elementary particles. Perhaps the simplest case is that of a scalar field $\psi : \mathbb{R}^n \to \mathbb{C} : x \mapsto \psi(x)$ for a scalar boson.

Physical interactions are formulated in terms of so-called Lagrangians. For a scalar boson we stipulate a Lagrangian of the form

$$\mathscr{L}_{\text{boson}}(\psi, \partial\psi) = \partial^\mu \psi^* \partial_\mu \psi - m^2 \psi^* \psi ,$$

in which $\partial^\mu = g^{\mu\nu} \partial_\nu$, with $g^{\mu\nu}$ the holor of the dual of a Lorentzian type metric tensor $\mathbf{G} = g_{\mu\nu} dx^\mu \otimes dx^\nu$, and in which $z^* \in \mathbb{C}$ denotes the complex conjugate of $z \in \mathbb{C}$.

(5) **a1.** Show that, if $x = x(\overline{x})$ denotes a coordinate transformation, then $\overline{\partial}_\mu = \dfrac{\partial x^\nu}{\partial \overline{x}^\mu}\partial_\nu$ and $\overline{\partial}^\mu = \dfrac{\partial \overline{x}^\mu}{\partial x^\nu}\partial^\nu$.

Definition (Scalar Field). Let $u : \mathbb{R}^n \to \mathbb{C} : x \mapsto u(x)$ be a scalar field. If $x = x(\overline{x})$ is a coordinate transformation, we set $\overline{u}(\overline{x}) \stackrel{\text{def}}{=} u(x)$.

(5) **a2.** Show that $\mathscr{L}_{\text{boson}}$ is invariant under coordinate transformations, i.e. $\mathscr{L}_{\text{boson}} = \overline{\mathscr{L}}_{\text{boson}}$.

(5) **b1.** Show that $\mathscr{L}_{\text{boson}}$ is invariant under *global* phase shifts of the type $\widetilde{\psi}(x) = e^{i\alpha}\psi(x)$, with $\alpha \in \mathbb{R}$ constant.

(5) **b2.** Show that $\mathscr{L}_{\text{boson}}$ is *not* invariant under *local* phase shifts, i.e. $\widetilde{\psi}(x) = e^{i\alpha(x)}\psi(x)$, in which now $\alpha \in C^\infty(\mathbb{R}^n)$ represents a smooth real-valued scalar function.

In order to render the theory invariant under local phase shifts, a remedy has been proposed that requires the introduction of a so-called *gauge field*, as follows. Instead of ordinary partial derivatives $\partial_\mu\psi$, consider covariant derivatives of the form $D^{\text{A}}_\mu\psi = \partial_\mu\psi + iqA_\mu\psi$, in which q is a "coupling constant" (a.k.a. "charge") and in which $A_\mu : \mathbb{R}^n \to \mathbb{C} : x \mapsto A_\mu(x)$ represents the (holor of the) gauge field.

To enforce the desired invariance we require $\widetilde{D}^{\text{A}}_\mu\widetilde{\psi}(x) = e^{i\alpha(x)}D^{\text{A}}_\mu\psi(x)$ given the local transformation $\widetilde{\psi}(x) = e^{i\alpha(x)}\psi(x)$ considered in b2. Here $\widetilde{D}^{\text{A}}_\mu = \partial_\mu + iq\widetilde{A}_\mu$ denotes the appropriately transformed covariant derivative.

(5) **c.** Show that local phase transformations $\widetilde{\psi} = e^{i\alpha}\psi$ induce gauge field transformations $\widetilde{A}_\mu = A_\mu - \frac{1}{q}\partial_\mu\alpha$.

If the gauge field has any dynamics of its own, it must be incorporated into the Lagrangian in a gauge invariant way. It is stipulated that this is accomplished by adding a term

$$\mathscr{L}_{\text{E.M.}}(A) = -\frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A)$$

to the foregoing Lagrangian (besides the replacement $\partial_\mu \to D^{\text{A}}_\mu$), in which

$$F_{\mu\nu}(A) \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

(5) **d.** Show that $F_{\mu\nu}(A)$ (and thus $\mathscr{L}_{\text{E.M.}}(A)$) is invariant under gauge field transformations, recall c.

☞ All in all, gauge invariance of the bosonic Lagrangian requires a gauge field in the form of the 'electro-magnetic' 4-vector potential A_μ, and the full gauge invariant Lagrangian becomes

$$\mathscr{L}(\psi, \partial\psi, A) = \mathscr{L}_{\text{boson}}(\psi, D^{\text{A}}\psi) + \mathscr{L}_{\text{E.M.}}(A) = D^\mu_{\text{A}}\psi^* D^{\text{A}}_\mu\psi - m^2\psi^*\psi - \frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A)\,.$$

A is the so-called electromagnetic 4-vector potential, a relativistic vector function combining an electric scalar potential ϕ and a magnetic 3-vector potential $\boldsymbol{A}$ into a single four-vector $A = (\phi/c, \boldsymbol{A})$. The EM-fields $(\boldsymbol{E}, \boldsymbol{B})$ follow from these potentials by

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \quad \text{respectively} \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}\,.$$

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(**35**) **3. Levi-Civita Connection & Covariant Derivative**[1]**.**

In this problem we consider a vector field $v(x) = v^i(x)\partial_i$ on an n-dimensional Riemannian manifold $\mathbb{M}$, locally decomposed relative to a coordinate basis $\{\partial_i = \partial/\partial x^i\}_{i=1,\ldots,n}$ at each basepoint $x \in \mathbb{M}$. The Riemannian metric tensor field is given by $G(x) = g_{ij}(x)dx^i \otimes dx^j$, with $\langle dx^i, \partial_j \rangle = \delta^i_j$.

We are interested in the rate of change of the vector field, $dv(x(t))/dt$, along a parametrized curve $x : \mathbb{R} \to \mathbb{M} : t \mapsto x(t)$. The local tangent at $x(t) \in \mathbb{M}$ is denoted by $\dot{x}(t) = \dot{x}^i(t)\partial_i$, with $\dot{} \equiv d/dt$.

A coordinate transformation $x = x(\overline{x})$ induces a local basis transformation, with transformed basis $\{\overline{\partial}_i = \partial/\partial\overline{x}^i\}_{i=1,\ldots,n}$ and Jacobian matrices

$$A^i_j(\overline{x}) = \frac{\partial x^i(\overline{x})}{\partial \overline{x}^j} \qquad \text{and} \qquad B^i_j(x) = \frac{\partial \overline{x}^i(x)}{\partial x^j}$$

Treat A^i_j and B^i_j implicitly as functions of $\overline{x}$, resp. x (as indicated explicitly in their definitions above).

($2\frac{1}{2}$) **a1.** Show that $v^i = A^i_j \overline{v}^j$ ('vector transformation').

($2\frac{1}{2}$) **a2.** Show that $\overline{g}_{ij} = A^k_i A^\ell_j g_{k\ell}$.

($2\frac{1}{2}$) **a3.** Show that $\overline{\partial}_i A^k_j = \overline{\partial}_j A^k_i$.

($2\frac{1}{2}$) **a4.** Show that $\overline{g}_{k\ell} B^\ell_m = g_{m\ell} A^\ell_k$.

Below we shall write v^i and $\overline{v}^i$ as shorthands for $v^i(x(t))$ and $\overline{v}^i(\overline{x}(t))$.

($2\frac{1}{2}$) **b.** Show that the acceleration components $\dot{v}^i = dv^i/dt$ do *not* obey the vector transformation law.

($2\frac{1}{2}$) **c.** Establish the relation between $\overline{\partial}_k \overline{g}_{ij}$ and $\partial_k g_{ij}$, likewise accounting for the role of the Jacobian A^i_j (and its derivatives). Do the partial derivatives $\partial_k g_{ij}$ constitute a tensor holor?

We define the Christoffel symbols of the first kind as follows: $\gamma_{ijk} \stackrel{\text{def}}{=} \frac{1}{2}\left(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik}\right)$.

(5) **d.** Prove: $\overline{\gamma}_{ijk} = A^\ell_i A^m_j A^n_k \gamma_{\ell mn} + B^\ell_m \overline{\partial}_k A^m_i \overline{g}_{j\ell}$.

(5) **e1.** Show that $\dot{v}^i = A^i_j \dot{\overline{v}}^j + \overline{\partial}_k A^i_m \dot{\overline{x}}^k \overline{v}^m$.

(5) **e2.** Use d to show that $\overline{\partial}_k A^i_m \dot{\overline{x}}^k = A^i_\ell \overline{\gamma}^\ell_{mk} \dot{\overline{x}}^k - A^\ell_m \gamma^i_{\ell k} \dot{x}^k$, in which $\gamma^k_{ij} \stackrel{\text{def}}{=} g^{k\ell}\gamma_{i\ell j}$.
[*Hint:* Contract the identity in d with $A^q_p \overline{g}^{jp} \dot{\overline{x}}^k$.]

(5) **f.** Use e1 and e2 to show that $\dfrac{Dv^i}{dt} \stackrel{\text{def}}{=} \dfrac{dv^i}{dt} + \gamma^i_{jk} v^j \dot{x}^k$ transforms as a vector.

The End

[1] Adapted from: H. Rund, The Differential Geometry of Finsler Spaces, Springer-Verlag, 1959, Chapter III, § 2.