EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

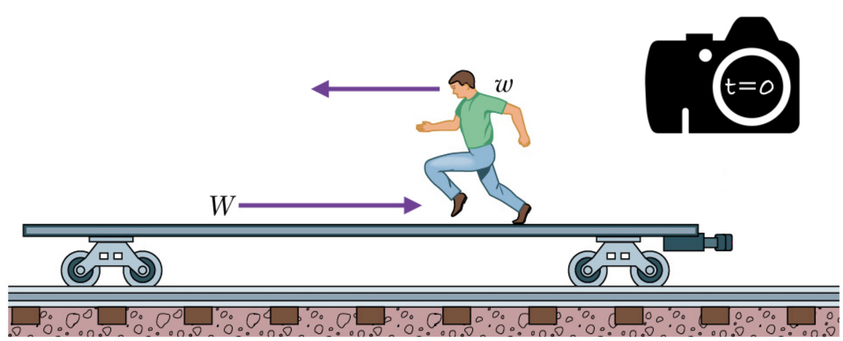
Course code: 2WAH0. Date: Thursday April 13, 2017. Time: 9h00–12h00. Place: Paviljoen L10.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- Einstein summation convention applies throughout for all repeated indices.
- You may consult an immaculate hardcopy of the online draft notes “Tensor Calculus and Differential Geometry (2WAH0)” by Luc Florack. No equipment may be used.

(**25**) **1.** GALINEAN BOOSTS

An observer, standing at the origin $O : (x, y, z) = (0, 0, 0)$ of a Cartesian coordinate system (x, y, z) attached to the ground, is witnessing a railroad flatcar passing by with velocity W along the x-axis. A freerunner is running on the deck with a horizontal velocity w *relative to the flatcar*, i.e. relative to the origin $O' : (x', y', z') = (0, 0, 0)$ of a comoving Cartesian coordinate system (x', y', z') attached to the deck, with x'-axis aligned with the x-axis. The sketch shows a snapshot at time $t = 0$, which is the moment at which the static and comoving coordinate origins coincide: $O = O'$.



MAN RUNNING WITH VELOCITY w RELATIVE TO THE DECK OF RAILROAD FLATCAR, WHICH IN TURN IS ROLLING ALONG A STRAIGHT HORIZONTAL TRACK WITH VELOCITY W RELATIVE TO THE GROUND. IN THIS PICTURE $w < 0$, $W > 0$.

Suppose $(w, 0, 0) = (-W, 0, 0)$. Gut feeling tells us that the freerunner is in rest relative to the external observer. A smart student, however, manages to embarrass her teacher with the following paradox:

"Since the velocity of the freerunner relative to the ground is the zero vector", she argues, "it must remain the zero vector relative to *any other* coordinate system, since a change of coordinate basis $(\partial_x, \partial_y, \partial_z) \to (\partial_{x'}, \partial_{y'}, \partial_{z'})$ induces a *linear* transformation of the components of a vector. Thus he cannot be moving relative to the deck either (i.e. $w = 0$), which in turn means that the flatcar must be standing still ($W = 0$), otherwise the freerunner would not be in rest relative to the external observer…"

(5) **a.** Her conclusion is clearly preposterous, but what is wrong with her 'linear transformation' argument?

The transformation relating the coordinate frames considered in the problem is of a *dynamic* nature, thus we have to include time as a fourth dimension. The relative motion implies the following relation between the respective Cartesian coordinates of an event in space and time:

$$\left\{ \begin{array}{lcl} \overline{x}^i & = & x^i - v^i t \\ \overline{t} & = & t \end{array} \right. \qquad \text{or, inversely,} \qquad \left\{ \begin{array}{lcl} x^i & = & \overline{x}^i + v^i \overline{t} \\ t & = & \overline{t} \end{array} \right.$$

Note that, if restricted to spatial components only (tacitly assumed by the student in her whimsical argument), this transformation is *nonlinear*. In particular we have for a 3-velocity $\dot{\overline{x}}^i = \dot{x}^i - v^i$t. The corresponding transformation of the 4-dimensional spacetime coordinate basis and its dual is given by

$$\left\{ \begin{array}{lcl} \overline{\partial}_i & = & \partial_i \\ \overline{\partial}_t & = & \partial_t + v^i \partial_i \end{array} \right. \qquad \text{resp.} \qquad \left\{ \begin{array}{lcl} dx^i & = & d\overline{x}^i + v^i d\overline{t} \\ dt & = & d\overline{t} \end{array} \right.$$

An object moving relative to the (x, y, z, t)-coordinate frame along a path $(x, y, z, t) = (x(t), y(t), z(t), t)$, say, has a geometric *4-velocity*, the (contravariant) components of which do transform linearly, viz. $\dot{x}^\mu \partial_\mu = \dot{x}^i \partial_i + \partial_t = \dot{\overline{x}}^i \overline{\partial}_i + \overline{\partial}_t = \dot{\overline{x}}^\mu \overline{\partial}_\mu$.

We embed the components of a given spatial '3-velocity' vector $v^i \partial_i = v^1 \partial_x + v^2 \partial_y + v^3 \partial_z$ into a spatiotemporal '4-velocity' $v^\mu \partial_\mu = v^0 \partial_t + v^1 \partial_x + v^2 \partial_y + v^3 \partial_z$. Here and henceforth Latin indices $i, j, k, \ldots = 1, 2, 3$ index spatial components, whereas Greek indices $\mu, \nu, \rho = 0, 1, 2, 3$ pertain to spatiotemporal components, with index value 0 reserved for time. The added temporal component of the 4-velocity is invariably defined by the (coordinate independent) constant $v^0 = 1$.

Let $\mathscr{P}$ be a spatiotemporal event with Cartesian spacetime coordinates $(x, y, z, t) \in \mathbb{R}^4$ relative to the ground frame, and coordinates $(\overline{x}, \overline{y}, \overline{z}, \overline{t}) \in \mathbb{R}^4$ relative to a frame moving to the right with constant horizontal 4-velocity $(v, 0, 0, 1)$.

(5) **b.** Express $\overline{x}^\mu$ in terms of x^ν and vice versa.

v.s.

(5) **c.** Express the coordinate basis vectors $\overline{\partial}_\mu$ in terms of ∂_ν.

v.s.

(5) **d.** Express the dual coordinate basis one-forms dx^μ in terms of $d\overline{x}^\nu$.

v.s.

(5) **e.** Show that a spatial 3-vector $v^i \partial_i$ does not obey a (linear) contravariant transformation law, but a suitably constructed 4-vector $v^\mu \partial_\mu$ does. Is the stipulated definition $v^0 = 1$ consistent?

v.s.

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(**40**) **2.** EUCLIDEAN GEODESICS

Geodesics in Euclidean 2-space, endowed with a flat connection, are most conveniently represented in terms of Cartesian coordinates, viz. if $\gamma : (x, y) = (x(t), y(t))$ is an affinely parametrized geodesic, then γ satisfies the following system of o.d.e.'s (with each dot representing a derivative d/dt)

$$* : \left\{ \begin{array}{lcl} \ddot{x} & = & 0 \\ \ddot{y} & = & 0\,. \end{array} \right.$$

We introduce polar coordinates $(r, \phi) \in [0, \infty) \times [0, 2\pi)$ in the usual way: $\star : \left\{ \begin{array}{lcl} x & = & r\cos\phi \\ y & = & r\sin\phi \end{array} \right. .$

(5) **a.** Derive the geodesic equations for $\gamma : (r, \phi) = (r(t), \phi(t))$ in polar coordinates by substituting the transformation $\star$ into the system $*$.

Straightforward substitution yields

$$** : \left\{ \begin{array}{lcl} \ddot{r}\cos\phi - 2\dot{r}\sin\phi\dot{\phi} - r\cos\phi\dot{\phi}^2 - r\sin\phi\ddot{\phi} & = & 0 \\ \ddot{r}\sin\phi + 2\dot{r}\cos\phi\dot{\phi} - r\sin\phi\dot{\phi}^2 + r\cos\phi\ddot{\phi} & = & 0\,. \end{array} \right.$$

(10) **b.** Compute all Christoffel symbols Γ^k_{ij} as well as all first order partial derivatives $\partial_\ell \Gamma^k_{ij}$ of the flat connection relative to polar coordinates.
[HINT: SHOW THAT THE ONLY NONZERO Γ-SYMBOLS ARE $\Gamma^\phi_{\phi r}$, $\Gamma^\phi_{r\phi}$ AND $\Gamma^r_{\phi\phi}$, AND THAT THESE DO NOT DEPEND ON ϕ.]

The only nonzero Christoffel symbols are $\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}$ and $\Gamma^r_{\phi\phi} = -r$. As a result the only nontrivial first order partial derivatives are $\partial_r \Gamma^r_{\phi\phi} = -1$, $\partial_r \Gamma^\phi_{r\phi} = \partial_r \Gamma^\phi_{\phi r} = -\frac{1}{r^2}$. Note that these symbols can also be obtained by inspection of the coefficients of the nonlinear terms in the next problem.

(5) **c.** Derive the geodesic equations for $\gamma : (r, \phi) = (r(t), \phi(t))$ from the Christoffel symbols obtained in b.

Using the Christoffel symbols from b in the geodesic equation $\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$ that is valid in any coordinate system yields

$$*** : \left\{ \begin{array}{lcl} \ddot{r} - r\dot{\phi}^2 & = & 0 \\ \ddot{\phi} + \dfrac{2}{r}\dot{r}\dot{\phi} & = & 0\,. \end{array} \right.$$

(5) **d.** Show that the equations obtained in a and c are consistent.

Multiply the first equation in $**$ by $\cos\phi$ and the second by $\sin\phi$, then add up. This yields the first equation of $***$. Multiply the first equation in $**$ by $\sin\phi$ and the second by $\cos\phi$, then subtract. This yields the second equation of $***$. Alternatively, rewrite $**$ as follows:

$$** : \left\{ \begin{array}{lcl} (\ddot{r} - r\dot{\phi}^2)\cos\phi - (\dfrac{2}{r}\dot{r}\dot{\phi} + \ddot{\phi})r\sin\phi & = & 0 \\ (\ddot{r} - r\dot{\phi}^2)\sin\phi + (\dfrac{2}{r}\dot{r}\dot{\phi} + \ddot{\phi})r\cos\phi & = & 0\,. \end{array} \right.$$

(5) **e.** Explain why a geodesic that does not pass through the origin $(x, y) = (0, 0)$ experiences a 'pseudo-acceleration' $\ddot{r} \neq 0$ for its radial coordinate, and show that this does not occur for a geodesic that does pass through the origin.

The 'pseudo-acceleration' $\ddot{r} = r\dot{\phi}^2$ is needed to compensate for the fact that the r-coordinate along a geodesic that does not pass through the origin $(x, y) = (0, 0)$ is not a linear function of x and y and thus not a Cartesian coordinate along the curve. If the geodesic does pass through the origin, then $\phi =$ constant (straight line), so $\dot{\phi} = 0$. Inserting this in $***$ yields (besides $\ddot{\phi} = 0$) $\ddot{r} = 0$, whence no 'pseudo-acceleration'.

For a Levi-Civita connection the covariant Riemann tensor is defined as follows: $R_{\ell ijk} = g_{\ell m} R^m{}_{ijk}$, in which

$$R^m{}_{ijk} = \partial_j \Gamma^m_{ik} - \partial_k \Gamma^m_{ij} + \Gamma^m_{nj}\Gamma^n_{ik} - \Gamma^m_{nk}\Gamma^n_{ij} \, .$$

Without proof we state that among all components of $R_{\ell ijk}$, with $i, j, k, \ell = 1, 2$, there is only one independent degree of freedom, which we may take to correspond to R_{1212}.

(5) **f1.** Expand the Einstein summation in the expression for $R_{r\phi r\phi}$ symbolically in terms of all *nonzero* Γ-terms *without substituting their actual functional values* (recall the hint in problem b).

The only nontrivial Γ-terms are $\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi}$ and $\Gamma^r_{\phi\phi}$, so that $R_{r\phi r\phi} = g_{rr}\left(\partial_r \Gamma^r_{\phi\phi} - \Gamma^r_{\phi\phi}\Gamma^\phi_{\phi r}\right)$.

(5) **f2.** Evaluate $R_{r\phi r\phi}$ by substituting the functional values for the Γ-symbols in terms of r and ϕ as obtained in problem b and explain your result.

Using the result of problem b, together with $g_{rr} = 1$, we obtain $R_{r\phi r\phi} = g_{rr}\left(\partial_r \Gamma^r_{\phi\phi} - \Gamma^r_{\phi\phi}\Gamma^\phi_{\phi r}\right) = \partial_r(-r) - (-r)\frac{1}{r} = 0$. This result is as expected, since (i) the Christoffel symbols of the flat connection vanish identically in Cartesian coordinates, whence $R_{\ell ijk} = 0$ in Cartesian coordinates $x \in \mathbb{R}^2$, and (ii) the holor of the Riemann tensor transforms linearly under coordinate transformations, thus $\overline{R}_{\ell ijk} = 0$ in any other coordinate system $\overline{x} \in \mathbb{R}^2$. (This argument is not quite rigorous, since polar coordinates $(\overline{x}^1, \overline{x}^2) = (r, \phi)$ are somewhat problematic. We ignore this subtlety here.)

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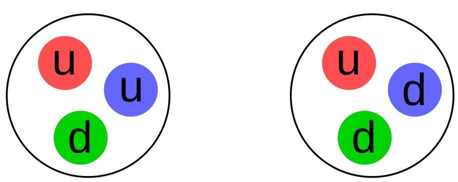
(35) 3. QUARKS & ISOSPIN MULTIPLETS



QUARK MODEL OF PROTON (LEFT) AND NEUTRON (RIGHT). THE SIGNIFICANCE OF COLOUR IS NOT ADDRESSED IN THIS PROBLEM.

According to a model introduced by Gell-Mann, quarks are the building blocks of elementary particles known as hadrons. We consider two 'flavours', referred to as up (u) and down quark (d). These two quarks $\{\mathsf{u}, \mathsf{d}\}$ are considered to span an abstract 2-dimensional complex inner product space $\mathbb{C}^2$ through the following identification:

$$\mathsf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \mathsf{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Given $v, w \in \mathbb{C}^2$, their standard complex inner product is given by $(v|w) = \overline{v}^1 w^1 + \overline{v}^2 w^2$, in which the overline denotes complex conjugation. The Gram matrix G with components g_{ij}, $i, j = 1, 2$, is defined by $(v|w) = g_{ij}\overline{v}^i w^j$. In particular we have $(\mathsf{u}|\mathsf{u}) = (\mathsf{d}|\mathsf{d}) = 1$, $(\mathsf{u}|\mathsf{d}) = \overline{(\mathsf{d}|\mathsf{u})} = 0$.

(5) **a.** Explain why we do not need to make any distinction between the 'contravariant' components of a vector $v \in \mathbb{C}^2$ relative to the basis $\{\mathsf{u}, \mathsf{d}\}$ and their covariant counterparts, i.e. the corresponding components of the covector $\hat{v} \doteq (v|\_) \in (\mathbb{C}^2)^*$.

Let $v = a\mathsf{u} + b\mathsf{d} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$ relative to basis $\{\mathsf{u}, \mathsf{d}\}$ for some $a, b \in \mathbb{C}$. Then $\hat{v} = \sharp v = a\sharp\mathsf{u} + b\sharp\mathsf{d}$ with $\sharp\mathsf{u} = g_{11}\hat{\mathsf{u}} + g_{12}\hat{\mathsf{d}} = \hat{\mathsf{u}}$ and similarly $\sharp\mathsf{d} = g_{21}\hat{\mathsf{u}} + g_{22}\hat{\mathsf{d}} = \hat{\mathsf{d}}$, so that $\hat{v} = a\hat{\mathsf{u}} + b\hat{\mathsf{d}} = (a, b) \in (\mathbb{C}^2)^*$ relative to the dual basis $\{\hat{\mathsf{u}}, \hat{\mathsf{d}}\}$. Thus a vector v has the same contravariant components relative to $\{\mathsf{u}, \mathsf{d}\}$ as its covariant counterpart $\hat{v}$ has relative to $\{\hat{\mathsf{u}}, \hat{\mathsf{d}}\}$: $v_1 = v^1 = a, v_2 = v^2 = b$.

(5) **b.** Show that if $U : \mathbb{C}^2 \to \mathbb{C}^2 : v \mapsto Uv$ preserves the inner product, i.e. if $(Uv|Uw) = (v|w)$ for all $v, w \in \mathbb{C}^2$, then the matrix U must be of the form

$$U = \begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix},$$

for some $\alpha, \beta \in \mathbb{C}$, with $\det U = |\alpha|^2 + |\beta|^2 = 1$, i.e. a unitary matrix.

From its definition it follows that $U^\dagger U = I$, the identity matrix. Without loss of generality we may set

$$U = \begin{pmatrix} \alpha & -B \\ \overline{\beta} & \overline{A} \end{pmatrix},$$

for some constants $\alpha, \beta, A, B \in \mathbb{C}$. The above matrix identity then yields three independent equations:

$$(*) \quad \begin{cases} |\alpha|^2 + |\beta|^2 &= 1, \\ |A|^2 + |B|^2 &= 1, \\ -\overline{\alpha}B + \overline{A}\beta &= 0. \end{cases}$$

By taking the determinant of the matrix identity we also observe that $1 = \det I = \det U^\dagger U$, i.e.

$$(**) \quad \alpha\overline{A} + \overline{\beta}B.$$

Multiplying the complex conjugate of the third equation of $(*)$ by B and subtracting A times $(**)$ yields, when combined with the second equation of $(*)$, $\beta = B$. Likewise, multiplying $(**)$ by A and subtracting B times the complex conjugate of the last equation of $(*)$ yields $\alpha = A$. This establishes the form of U, subject to the remaining constraint $|\alpha|^2 + |\beta|^2 = 1$, i.e. the first equation of $(*)$.

Consistent with experimental observations, the laws of nature governing nuclear forces appear to be symmetric under unitary transformations in the space $\mathbb{C}^2$ spanned by the up and down quark. However, up and down quarks have never been observed in isolation. We therefore consider a subclass of hadrons consisting of three quarks, the so-called baryons, to which the familiar proton and neutron belong.

To represent quark triplets we consider the threefold tensor product space $\mathbf{T}^3_0(\mathbb{C}^2) = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

(5) **c.** Show that $\dim \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = 8$, and provide a basis of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ induced by $\{\mathsf{u}, \mathsf{d}\}$.

A naturally induced basis is given by $\{\mathsf{u}\otimes\mathsf{u}\otimes\mathsf{u}, \mathsf{u}\otimes\mathsf{u}\otimes\mathsf{d}, \mathsf{u}\otimes\mathsf{d}\otimes\mathsf{u}, \mathsf{d}\otimes\mathsf{u}\otimes\mathsf{u}, \mathsf{u}\otimes\mathsf{d}\otimes\mathsf{d}, \mathsf{d}\otimes\mathsf{u}\otimes\mathsf{d}, \mathsf{d}\otimes\mathsf{d}\otimes\mathsf{u}, \mathsf{d}\otimes\mathsf{d}\otimes\mathsf{d}\}$. Indeed, since $\dim \mathbb{C}^2 = 2$ its follows that $\dim \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = 2^3 = 8$.

It turns out that there are four so-called Δ-baryons of equal mass. This suggests that we consider a four-dimensional subspace invariant under unitary transformations.

(5) **d.** Show that the subspace $\bigvee^3(\mathbb{C}^2) = \mathbb{C}^2 \otimes_s \mathbb{C}^2 \otimes_s \mathbb{C}^2 \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbf{T}^3_0(\mathbb{C}^2)$ of fully symmetric 3-tensors is indeed four-dimensional, and provide a basis induced by $\{\mathsf{u}, \mathsf{d}\}$.

We have, with dimension $n = \dim \mathbb{C}^2 = 2$ and rank $d = 3$, $\dim \bigvee^3(\mathbb{C}^2) = \begin{pmatrix} n+d-1 \\ d \end{pmatrix} = 4$. A naturally induced basis is given by $\{\mathsf{u}\otimes\mathsf{u}\otimes\mathsf{u}, \mathsf{u}\otimes\mathsf{u}\otimes\mathsf{d}+\mathsf{u}\otimes\mathsf{d}\otimes\mathsf{u}+\mathsf{d}\otimes\mathsf{u}\otimes\mathsf{u}, \mathsf{u}\otimes\mathsf{d}\otimes\mathsf{d}+\mathsf{d}\otimes\mathsf{u}\otimes\mathsf{d}+\mathsf{d}\otimes\mathsf{d}\otimes\mathsf{u}, \mathsf{d}\otimes\mathsf{d}\otimes\mathsf{d}\}$. Alternatively, using the symmetric tensor product operator $\vee$ we may write this as $\{\mathsf{u}\vee\mathsf{u}\vee\mathsf{u}, \mathsf{u}\vee\mathsf{u}\vee\mathsf{d}, \mathsf{u}\vee\mathsf{d}\vee\mathsf{d}, \mathsf{d}\vee\mathsf{d}\vee\mathsf{d}\}$.

The four Δ-baryons are Δ^{++}, Δ^{+}, Δ^{0}, Δ^{-}, in which the superscript indicates elementary electric charge $Q(\Delta^{++}) = 2\mathrm{e}$, $Q(\Delta^{+}) = \mathrm{e}$, $Q(\Delta^{0}) = 0$, respectively $Q(\Delta^{-}) = -\mathrm{e}$.

In view of the observed electric charges we assume that the quarks $\mathsf{u}, \mathsf{d} \in \mathbb{C}^2$ each have a definite charge, $Q(\mathsf{u})$, respectively $Q(\mathsf{d})$, with $Q(\mathsf{u}) \geq Q(\mathsf{d})$ to resolve ambiguity.

(5) **e.** Determine the elementary charges $Q(\mathsf{u})$ and $Q(\mathsf{d})$ of the u and d quark, such that the Δ-baryons can be consistently related to the basis of $\bigvee^3(\mathbb{C}^2)$ (recall d), and explain assumptions you have made.

We have $Q(\mathsf{u}\vee\mathsf{u}\vee\mathsf{u}) = 3Q(\mathsf{u})$, $Q(\mathsf{u}\vee\mathsf{u}\vee\mathsf{d}) = 2Q(\mathsf{u})+Q(\mathsf{d})$, $Q(\mathsf{u}\vee\mathsf{d}\vee\mathsf{d}) = Q(\mathsf{u})+2Q(\mathsf{d})$, $Q(\mathsf{d}\vee\mathsf{d}\vee\mathsf{d}) = 3Q(\mathsf{d})$. The assumption made here is that Q is well-defined for linear combinations Δ of tensor products of the form $q_1\otimes q_2\otimes q_3 \in \mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2$, with $q_i \in \{\mathsf{u}, \mathsf{d}\}$, having an equal number of u's and d's. In this case, $Q(\Delta) = Q(q_1\otimes q_2\otimes q_3) = n_u Q(\mathsf{u}) + n_d Q(\mathsf{d})$, in which n_u and $n_d = 3 - n_u$ indicate the respective numbers of u and d quarks among (q_1, q_2, q_3). With the choice $Q(\mathsf{u}) \geq Q(\mathsf{d})$ we must apparently set $3Q(\mathsf{u}) = 2\mathrm{e}$, whence $Q(\mathsf{u}) = \frac{2}{3}\mathrm{e}$, from which it follows consistently that $Q(\mathsf{d}) = -\frac{1}{3}\mathrm{e}$. Thus we have the identifications $\Delta^{++} \sim \mathsf{u}\vee\mathsf{u}\vee\mathsf{u} = \mathsf{u}\otimes\mathsf{u}\otimes\mathsf{u}$, $\Delta^{+} \sim \mathsf{u}\vee\mathsf{u}\vee\mathsf{d} = \mathsf{u}\otimes\mathsf{u}\otimes\mathsf{d}+\mathsf{u}\otimes\mathsf{d}\otimes\mathsf{u}+\mathsf{d}\otimes\mathsf{u}\otimes\mathsf{u}$, $\Delta^{0} \sim \mathsf{u}\vee\mathsf{d}\vee\mathsf{d} = \mathsf{u}\otimes\mathsf{d}\otimes\mathsf{d}+\mathsf{d}\otimes\mathsf{u}\otimes\mathsf{d}+\mathsf{d}\otimes\mathsf{d}\otimes\mathsf{u}$, and $\Delta^{-} \sim \mathsf{d}\vee\mathsf{d}\vee\mathsf{d} = \mathsf{d}\otimes\mathsf{d}\otimes\mathsf{d}$.

Recall $\dim \mathbb{C}^2\otimes_{\mathrm{S}}\mathbb{C}^2\otimes_{\mathrm{S}}\mathbb{C}^2 = 4$ while $\dim \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2 = 8$. We wish to capture the 'missing dimensions' by considering complementary subspaces. To this end consider $\mathbb{C}^2\otimes_{\mathrm{A}}\mathbb{C}^2\otimes\mathbb{C}^2 \subset \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ and $\mathbb{C}^2\otimes\mathbb{C}^2\otimes_{\mathrm{A}}\mathbb{C}^2 \subset \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, defined as the linear subspaces of all 3-tensors in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ that are antisymmetric with respect to their first two, respectively last two arguments.

(5) **f.** Show that $\dim \mathbb{C}^2\otimes_{\mathrm{A}}\mathbb{C}^2\otimes\mathbb{C}^2 = \dim \mathbb{C}^2\otimes\mathbb{C}^2\otimes_{\mathrm{A}}\mathbb{C}^2 = 2$, and provide a suitable basis for each induced by $\{\mathsf{u}, \mathsf{d}\}$.

A basis of $\mathbb{C}^2\otimes_{\mathrm{A}}\mathbb{C}^2\otimes\mathbb{C}^2$ is given by $\{(\mathsf{u}\otimes\mathsf{d}-\mathsf{d}\otimes\mathsf{u})\otimes\mathsf{u}, (\mathsf{u}\otimes\mathsf{d}-\mathsf{d}\otimes\mathsf{u})\otimes\mathsf{d}\}$. A basis of $\mathbb{C}^2\otimes\mathbb{C}^2\otimes_{\mathrm{A}}\mathbb{C}^2$ is given by $\{\mathsf{u}\otimes(\mathsf{u}\otimes\mathsf{d}-\mathsf{d}\otimes\mathsf{u}), \mathsf{d}\otimes(\mathsf{u}\otimes\mathsf{d}-\mathsf{d}\otimes\mathsf{u})\}$.

As with the Δ-baryons there is a similar approximate symmetry in the laws of nature governing the nuclear forces involving protons and neutrons.

(5) **g.** Assuming that the proton-neutron doublet constitutes a two-dimensional space spanned by $\{\mathsf{p}, \mathsf{n}\}$ that can be identified with one of the aforementioned subspaces, $\mathbb{C}^2\otimes_{\mathrm{A}}\mathbb{C}^2\otimes\mathbb{C}^2$, say, which identification would you make between the basis vectors p and n and those of $\mathbb{C}^2\otimes_{\mathrm{A}}\mathbb{C}^2\otimes\mathbb{C}^2$?

The basis $\{(\mathsf{u}\otimes\mathsf{d}-\mathsf{d}\otimes\mathsf{u})\otimes\mathsf{u}, (\mathsf{u}\otimes\mathsf{d}-\mathsf{d}\otimes\mathsf{u})\otimes\mathsf{d}\}$ of $\mathbb{C}^2\otimes_{\mathrm{A}}\mathbb{C}^2\otimes\mathbb{C}^2$ can be identified with $\{\mathsf{p}, \mathsf{n}\}$ through the identification $\mathsf{p} = (\mathsf{u}\otimes\mathsf{d}-\mathsf{d}\otimes\mathsf{u})\otimes\mathsf{u}$ and $\mathsf{n} = (\mathsf{u}\otimes\mathsf{d}-\mathsf{d}\otimes\mathsf{u})\otimes\mathsf{d}$, so that the actual charges $e = Q(\mathsf{p}) = 2Q(\mathsf{u}) + Q(\mathsf{d}) = 2\times\frac{2}{3}\mathrm{e} - \frac{1}{3}\mathrm{e}$ and $0 = Q(\mathsf{n}) = Q(\mathsf{u}) - 2Q(\mathsf{d}) = \frac{2}{3}\mathrm{e} - 2\times\frac{1}{3}\mathrm{e}$ are consistent with the quark charges and the additive rule established in problem e.