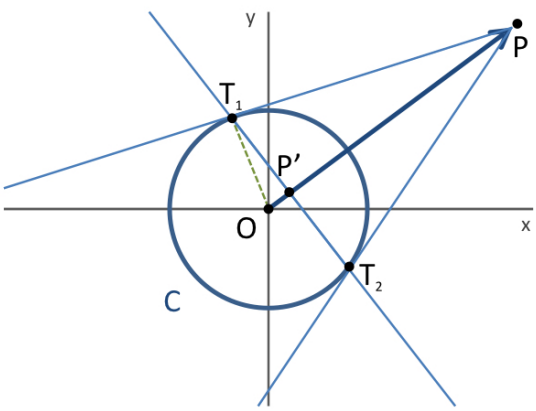
EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0-2WAH1. Date: Tuesday June 21, 2016. Time: 18h00–21h00. Place: AUD 11.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- Einstein summation convention applies throughout for all repeated indices.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No equipment may be used.



INDICATRIX $C : x^2 + y^2 = 1$. THE POINT P′ IS THE INVERSION OF THE POINT P IN THE CIRCLE C.

(**30**) **1.** INDICATRIX

The so-called indicatrix C : $x^2 + y^2 = 1$ is a geometrical construction of the Euclidean metric (standard inner product) in the Cartesian coordinate plane. Assume point P $= (p, q)$ is given, somewhere outside C. There are two lines through P tangent to C, touching at the points T_1, respectively T_2. The so-called polar line T_1T_2 intersects the line OP at point P′ $= (m, n)$, say.

(5) **a1.** Show that OP : $1 = 1$: OP′.
[HINT: COMPARE THE TRIANGLES P′TO AND TOP, WITH T EITHER T_1 OR T_2.]

The triangles P′TO and TOP (with T either T_1 or T_2) are similar, thus ratios of corresponding sides are equal. In particular we have OP : OT = OT : OP′, with OT = 1, the unit radius of C.

(5) **a2.** Show that the coordinates of P′ are given by $(m,n) = (\dfrac{p}{p^2+q^2}, \dfrac{q}{p^2+q^2})$.

The point P′ lies, by definition, on OP, so that $\overrightarrow{\mathrm{OP'}} \propto \overrightarrow{\mathrm{OP}} = (p,q)$, say $\overrightarrow{\mathrm{OP'}} = \alpha\,\overrightarrow{\mathrm{OP}}$. The proportionality constant $\alpha > 0$ follows from a1, viz. $\alpha = \mathrm{OP'}/\mathrm{OP} = 1/\mathrm{OP}^2 = 1/(p^2+q^2)$.

(5) **b.** Argue why $\overrightarrow{\mathrm{OP}} \perp \mathrm{T_1T_2}$, and show that the line $\mathrm{T_1T_2}$ is given by $px + qy = 1$.

A symmetry argument: The triangles $\mathrm{P'T_1P}$ and $\mathrm{P'T_2P}$ are eachother's mirror images in the line OP, thus in particular it follows that $\angle\mathrm{T_1P'P} = \angle\mathrm{T_2P'P} = 180°/2 = 90°$, i.e. $\overrightarrow{\mathrm{OP}} \perp \mathrm{T_1T_2}$. As a result, the line $\mathrm{T_1T_2}$ is given by $px + qy =$ constant. Substituting the coordinates of $\mathrm{P'} \in \mathrm{T_1T_2}$ from a2 yields constant $= 1$.

In summary, if $(m^1, m^2) = (m,n)$ and $(p^1,p^2) = (p,q)$ are the coordinates of $\overrightarrow{\mathrm{OP'}}$, respectively $\overrightarrow{\mathrm{OP}}$, $(x^1,x^2) = (x,y)$ any point on the indicatrix $\mathrm{C} : x^2+y^2 = 1$, and η_{ij}, $i,j = 1,2$, the matrix elements of the 2×2-identity matrix I, then, with $\cdot$ denoting the standard inner product on $\mathbb{R}^2$, we have

$$(\star)\qquad \left\{\begin{array}{rrcl} \overrightarrow{\mathrm{OP'}}\cdot\overrightarrow{\mathrm{OP}}: & mp+nq & = & 1\,, \\ \mathrm{T_1T_2}: & px+qy & = & 1\,, \\ \mathrm{C}: & x^2+y^2 & = & 1\,. \end{array}\right. \iff \left\{\begin{array}{rrcrcl} \overrightarrow{\mathrm{OP'}}\cdot\overrightarrow{\mathrm{OP}}: & \eta_{ij}m^ip^j & = & m_jp^j & = & 1\,, \\ \mathrm{T_1T_2}: & \eta_{ij}p^ix^j & = & p_jx^j & = & 1\,, \\ \mathrm{C}: & \eta_{ij}x^ix^j & = & x_jx^j & = & 1\,. \end{array}\right.$$

The indicatrix C is a pictorial representation of the covariant Euclidean metric tensor $H = \eta_{ij}dx^i \otimes dx^j$. The polar line $\mathrm{T_1T_2}$ provides a pictorial representation of the covector $\widehat{p} = p_i dx^i$ associated with the vector $\vec{p} = p^i\partial_i$ in the sense that it is just the differential of the left hand side of its defining equation. Note that the components p_i of the normal of the polar line happen to coincide numerically with the vector components p^i of $\vec{p} = \overrightarrow{\mathrm{OP}}$. This is an artefact of our choice of coordinates, or, alternatively, of the isotropic nature of C expressed in these coordinates, with $\eta_{ij} = 1$ if $i = j$ and zero otherwise.

Consider a linear change of coordinates, $x^i = L^i_j\overline{x}^j$, say, in which L^i_j, $i,j = 1,2$, are the matrix entries of the transformation matrix

$$L = \begin{pmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

(10) **c.** Rewrite the system $(\star)$ for $\overrightarrow{\mathrm{OP'}}\cdot\overrightarrow{\mathrm{OP}}$, $\mathrm{T_1T_2}$ and C, in terms of the new coordinates $(\overline{x}^1,\overline{x}^2) = (\overline{x},\overline{y})$, the parameters m,n,p,q, and the components a,b,c,d of the transformation matrix L.

Since all defining equations are represented in scalar form in $(\star)$ we have, after transformation:

$$(\star)\qquad \left\{\begin{array}{rrcrcl} \overrightarrow{\mathrm{OP'}}\cdot\overrightarrow{\mathrm{OP}}: & \overline{\eta}_{ij}\overline{m}^i\overline{p}^j & = & \overline{m}_j\overline{p}^j & = & 1\,, \\ \mathrm{T_1T_2}: & \overline{\eta}_{ij}\overline{p}^i\overline{x}^j & = & \overline{p}_j\overline{x}^j & = & 1\,, \\ \mathrm{C}: & \overline{\eta}_{ij}\overline{x}^i\overline{x}^j & = & \overline{x}_j\overline{x}^j & = & 1\,, \end{array}\right.$$

with

$$\overline{\eta}_{k\ell} = L^i_k\eta_{ij}L^j_\ell \quad m^i = L^i_j\overline{m}^j \quad p^i = L^i_j\overline{p}^j \quad x^i = L^i_j\overline{x}^j \quad \overline{m}_i = L^j_i m_j \quad \overline{p}_i = L^j_i p_j \quad \overline{x}_i = L^j_i x_j\,.$$

Explicitly in terms of the given parameters:

$$(\star)\qquad \left\{\begin{array}{rccl} \overrightarrow{\mathrm{OP'}}\cdot\overrightarrow{\mathrm{OP}}: & mp+nq & = & 1\,, \\ \mathrm{T_1T_2}: & (ap+bq)\overline{x}+(cp+dq)\overline{y} & = & 1\,, \\ \mathrm{C}: & (a^2+b^2)\overline{x}^2+2(ac+bd)\overline{x}\overline{y}+(c^2+d^2)\overline{y}^2 & = & 1\,. \end{array}\right.$$

(5) **d.** Suppose we would redraw the aforementioned geometric construct (point P, tangent cords $\mathrm{PT_1}$ and

PT_2, polar line $\mathrm{T}_1\mathrm{T}_2$, spherical reflection P', and indicatrix C, with all objects defined in terms of their $(\overline{x},\overline{y})$-coordinatizations, in a *rectangular* $(\overline{x},\overline{y})$-coordinate system in which basis (co)vectors are depicted as horizontal/vertical unit $\mathbb{R}^2$-coordinate (co)vectors:

$$\partial_{\overline{x}} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \partial_{\overline{y}} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Would the polar line $\mathrm{T}_1\mathrm{T}_2$ still be perpendicular to the vector $\overrightarrow{\mathrm{OP}}$? Motivate your answer by an explicit computation of the coordinates $\overline{p}^i$ and $\overline{p}_i$.

No. The inner product is now given by the Gram matrix

$$\overline{H} = L^{\mathrm{T}} L = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} .$$

Consequently, the vector $\overrightarrow{\mathrm{OP}}$ has coordinates $\overline{p}^i = (L^{-1})^i_j p^j$, whereas $\sharp\overrightarrow{\mathrm{OP}}$, the normal to $\mathrm{T}_1\mathrm{T}_2$, has coordinates $\overline{p}_i = L^j_i p_j$, i.e.

$$\overrightarrow{\mathrm{OP}} = \frac{1}{ad-bc}\begin{pmatrix} pd-cq \\ -bp+aq \end{pmatrix} \quad \sharp\overrightarrow{\mathrm{OP}} = \begin{pmatrix} ap+bq \\ cp+dq \end{pmatrix}$$

relative to bases $\{\partial_{\overline{x}}, \partial_{\overline{y}}\}$, respectively $\{d\overline{x}, d\overline{y}\}$. Drawn as vectors in the $\mathbb{R}^2$-plane with these basis (co)vectors represented as horizontal and vertical unit (co)vectors, these are, in general, not collinear. More precisely, we have

$$\cos\angle(\mathrm{T}_1\mathrm{T}_2, \overrightarrow{\mathrm{OP}}) = \frac{\overrightarrow{\mathrm{OP}} \cdot \sharp\overrightarrow{\mathrm{OP}}}{\|\mathrm{OP}\|\|\sharp\overrightarrow{\mathrm{OP}}\|} \neq 0 \text{ generically.}$$

(**25**) **2.** KILLING VECTORS

The Lie derivative w.r.t. a given vector field $\mathbf{v}$ is a linear first order partial differential operator satisfying the product rule, such that

$$\mathscr{L}_{\mathbf{v}} f = \mathbf{v} f \quad \text{and} \quad \mathscr{L}_{\mathbf{v}} \hat{\boldsymbol{\omega}} = d\left(\hat{\boldsymbol{\omega}}(\mathbf{v})\right) ,$$

for a scalar field f, respectively a covector field $\hat{\boldsymbol{\omega}}$.

(5) **a.** Show that $\mathscr{L}_{\mathbf{v}} dx^i = \partial_k v^i dx^k$.

We have $\mathscr{L}_{\mathbf{v}} dx^i = d(dx^i(v)) = dv^i = \partial_k v^i dx^k$.

Let $g = g_{ij} dx^i \otimes dx^j$ denote the covariant metric tensor field. By D_i we denote the covariant derivative operator w.r.t. x^i induced by the Levi-Civita connection.

(5) **b1.** Show that $\mathscr{L}_{\mathbf{v}} g = (v^k \partial_k g_{ij} + \partial_i v^k g_{kj} + \partial_j v^k g_{ik})\, dx^i \otimes dx^j$.

By virtue of the product rule we have

$$\mathscr{L}_{\mathbf{v}} g = \mathscr{L}_{\mathbf{v}}(g_{ij} dx^i \otimes dx^j) = \mathscr{L}_{\mathbf{v}} g_{ij} dx^i \otimes dx^j + g_{ij} \mathscr{L}_{\mathbf{v}} dx^i \otimes dx^j + g_{ij} dx^i \otimes \mathscr{L}_{\mathbf{v}} dx^j .$$

According to the definitions (and using a above) we have $\mathscr{L}_{\mathbf{v}} g_{ij} = v^k \partial_k g_{ij}$, $\mathscr{L}_{\mathbf{v}} dx^i = \partial_k v^i dx^k$, and $\mathscr{L}_{\mathbf{v}} dx^j = \partial_k v^j dx^k$. With some trivial dummy index relabelling this yields

$$\mathscr{L}_{\mathbf{v}} g = (v^k \partial_k g_{ij} + \partial_i v^k g_{kj} + \partial_j v^k g_{ik})\, dx^i \otimes dx^j .$$

(5) **b2.** Show that, equivalently, $\mathscr{L}_{\mathbf{v}} g = (D_i v_j + D_j v_i)\, dx^i \otimes dx^j$.

Substitute $g_{ik}\partial_j v^k = \partial_j(g_{ik}v^k) - \partial_j g_{ik} v^k = \partial_j v_i - \partial_j g_{ik} v^k$ and $g_{kj}\partial_i v^k = \partial_i(g_{kj}v^k) - \partial_i g_{kj} v^k = \partial_i v_j - \partial_i g_{kj} v^k$ and $v^k = g^{k\ell} v_\ell$ into the result of b1:

$$v^k \partial_k g_{ij} + \partial_i v^k g_{kj} + \partial_j v^k g_{ik} = \partial_j v_i + \partial_i v_j + g^{k\ell}(\partial_k g_{ij} - \partial_j g_{ik} - \partial_i g_{kj}) v_\ell \,,$$

and one recognizes on the r.h.s.

$$\partial_j v_i - \frac{1}{2} g^{k\ell}(-\partial_k g_{ij} + \partial_j g_{ik} + \partial_i g_{kj}) v_\ell = \partial_j v_i - \Gamma^\ell_{ij} v_\ell = D_j v_i \quad \text{and} \quad \partial_i v_j - \frac{1}{2} g^{k\ell}(-\partial_k g_{ij} + \partial_j g_{ik} + \partial_i g_{kj}) v_\ell = \partial_i v_j - \Gamma^\ell_{ji} v_\ell = D_i v_j \,.$$

A vector field $\mathbf{v}$ is called a Killing vector field if $\mathscr{L}_{\mathbf{v}} g = 0$. We henceforth consider the Euclidean metric of flat 3-dimensional space in Cartesian coordinates $(x^1, x^2, x^3) = (x, y, z)$, and a Killing vector field with component functions $(v^1, v^2, v^3) = (u, v, w)$.

(10) **c.** Show that the Killing vector field must satisfy $u_x = v_y = w_z = 0$ and $v_x + u_y = w_x + u_z = w_y + v_z = 0$.

In formal matrix notation we have, most easily using b2 and the observation that $D_i = \partial_i$ since $\Gamma^\ell_{ij} = 0$,

$$\mathscr{L}_{\mathbf{v}} g = \mathrm{tr}\left[\begin{pmatrix} 2u_x & v_x + u_y & w_x + u_z \\ v_x + u_y & 2v_y & w_y + v_z \\ w_x + u_z & w_y + v_z & 2w_z \end{pmatrix} \begin{pmatrix} dx \otimes dx & dx \otimes dy & dx \otimes dz \\ dy \otimes dx & dy \otimes dy & dy \otimes dz \\ dz \otimes dx & dz \otimes dy & dz \otimes dz \end{pmatrix} \right] ,$$

which vanishes identically iff the first matrix factor in this expression vanishes identically.

(**45**) **3.** LINEARIZED GRAVITY

In the presence of weak gravitational fields it is convenient to linearize the Einstein field equation for the gravitational potential, i.e. the pseudo-Riemannian spacetime metric tensor of General Relativity (GR):

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \,.$$

Here, $\eta_{\mu\nu}$ is the holor of the Minkowski metric of Special Relativity (SR), in an "inertial" coordinate system given by the universally constant matrix $\eta_{\mu\nu} \widehat{=} \mathrm{diag}\,\{-1, 1, 1, 1\}$, with $\mu, \nu = 0, 1, 2, 3$. Index value 0 refers to the temporal basis vector ∂_t, index values $1, 2, 3$ to the three independent spatial basis vectors $\partial_x, \partial_y, \partial_z$.

In linearized gravity one thinks of the perturbation $h_{\mu\nu}$ as a spatiotemporal tensor field propagating on the constant Minkowski background spacetime. Its entries are spatiotemporal functions with values that are small relative to those of the background field: $|h_{\mu\nu}| \ll 1$. This allows one to approximate exact GR laws in terms of expansions to order $\mathcal{O}(h)$, neglecting higher order terms. This $\mathcal{O}(h)$ approximation is assumed throughout this problem without explicit indication of discarded higher order terms.

The (exact) Einstein's field equation is a tensorial nonlinear partial differential equation that governs the spacetime metric $g_{\mu\nu}$ in relation to the flux of energy and momentum, which is captured by the energy-momentum tensor field $T_{\mu\nu}$:

$$(*) \qquad R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi\, T_{\mu\nu} \,.$$

(Units are defined such that $G = c = 1$, with G: gravitational constant, c: velocity of light.) Recall that the Ricci tensor is given in terms of the Levi-Civita Christoffel symbols by

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\beta_{\mu\alpha}\, \Gamma^\alpha_{\beta\nu} + \Gamma^\beta_{\mu\nu}\, \Gamma^\alpha_{\beta\alpha} \,.$$

An index preceded by a comma indicates a differentiation index: $f_{,\alpha} \stackrel{\text{def}}{=} \partial_\alpha f$.

(5) **a.** Show that $(*)$ reduces to $R_{\mu\nu} = 0$ in the homogeneous ("vacuum") case in which $T_{\mu\nu} = 0$.

If $T_{\mu\nu} = 0$, take the trace of $(*)$: $0 = g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}) = R - 2R = -R$, thus the vacuum field equation reduces to $R_{\mu\nu} = 0$.

Even this simplified vacuum field equation is a nonlinear partial differential equation for $g_{\mu\nu}$ that is not easily solved. Perturbation theory allows us to linearize $(*)$ and a fortiori its homogeneous counterpart.

(5) **b.** Show that, in linearized gravity, we may approximate $R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu}$, ignoring the $\Gamma\Gamma$-terms.

Making use of the constancy of $\eta_{\mu\nu}$, the connection coefficients are seen to be $\mathcal{O}(h)$:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(-\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} + \partial_\mu g_{\lambda\nu}) = \frac{1}{2}(\eta^{\lambda\alpha} + \mathcal{O}(h))(-\partial_\lambda h_{\mu\nu} + \partial_\nu h_{\mu\lambda} + \partial_\mu h_{\lambda\nu}) = \mathcal{O}(h)\,.$$

(5) **c.** Show that, in linearized gravity, $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\left(-\partial^\mu h_{\alpha\beta} + \partial_\beta h^\mu_\alpha + \partial_\alpha h^\mu_\beta\right)$. Explain why index raising in this expression may be carried out with the help of the background (dual) metric $\eta^{\mu\nu}$ instead of $g^{\mu\nu}$.

This follows from the previous derivation in b, ignoring the higher order terms, and setting $\eta^{\lambda\alpha}\partial_\lambda = \partial^\alpha$. This index raising by the Minkowski dual metric is allowed, since for the actual metric we have $g^{\lambda\alpha} = \eta^{\lambda\alpha} + \mathcal{O}(h)$, and the $\mathcal{O}(h)$-terms in this identity only contribute to the negligible $\mathcal{O}(h^2)$-terms in the Γ-symbols.

(5) **d.** Show that, in linearized gravity, $R_{\mu\nu} = \frac{1}{2}\left(-\partial^\alpha\partial_\alpha h_{\mu\nu} + \partial_\mu\partial_\alpha h^\alpha_\nu + \partial_\alpha\partial_\nu h^\alpha_\mu - \partial_\mu\partial_\nu h^\alpha_\alpha\right)$.

Use the definition of the Ricci tensor and the observations in b and c. We have (with the abbreviated notation $f_{,\gamma} = \partial_\gamma f$)

$$\Gamma^\mu_{\alpha\beta,\gamma} = \frac{1}{2}\left(-\partial^\mu\partial_\gamma h_{\alpha\beta} + \partial_\alpha\partial_\gamma h^\mu_\beta + \partial_\beta\partial_\gamma h^\mu_\alpha\right)\,,$$

and thus

$$\Gamma^\alpha_{\mu\nu,\alpha} = \frac{1}{2}\left(-\partial^\alpha\partial_\alpha h_{\mu\nu} + \partial_\mu\partial_\alpha h^\alpha_\nu + \partial_\nu\partial_\alpha h^\alpha_\mu\right)\,,$$

respectively

$$\Gamma^\alpha_{\mu\alpha,\nu} = \frac{1}{2}\left(-\partial^\alpha\partial_\nu h_{\mu\alpha} + \partial_\mu\partial_\nu h^\alpha_\alpha + \partial_\alpha\partial_\nu h^\alpha_\mu\right)\,.$$

Subtraction of the last two identities yields

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} = \frac{1}{2}\left(-\partial^\alpha\partial_\alpha h_{\mu\nu} + \partial_\mu\partial_\alpha h^\alpha_\nu + \partial^\alpha\partial_\nu h_{\mu\alpha} - \partial_\mu\partial_\nu h^\alpha_\alpha\right)\,.$$

Note that $\partial^\alpha\partial_\nu h_{\mu\alpha}$ may be rewritten as $\partial_\alpha\partial_\nu h^\alpha_\mu$.

One commonly abbreviates

$$h \stackrel{\text{def}}{=} h^\alpha_\alpha \quad \text{and} \quad \Box \stackrel{\text{def}}{=} \partial^\alpha\partial_\alpha\,,$$

so that

$$R_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} + \partial_\mu\partial_\alpha h^\alpha_\nu + \partial_\alpha\partial_\nu h^\alpha_\mu - \partial_\mu\partial_\nu h\right)\,.$$

Moreover, the Ricci scalar is defined as

$$R = R^\mu_\mu\,.$$

(5) **e.** Show that, in linearized gravity, $R = \partial_{\mu\alpha}h^{\mu\alpha} - \Box h$.

The main observation is that $R = g^{\mu\nu} R_{\mu\nu} = (\eta^{\mu\nu} + \mathcal{O}(h)) R_{\mu\nu}$, in which once again we may ignore the $\mathcal{O}(h)$-terms, since the Ricci tensor is itself $\mathcal{O}(h)$. Thus, using d, with the simplified notation for the trace and d'Alembertian,

$$R = \eta^{\mu\nu} R_{\mu\nu} = \frac{1}{2}\eta^{\mu\nu}\left(-\Box h_{\mu\nu} + \partial_\mu \partial_\alpha h^\alpha_\nu + \partial_\alpha \partial_\nu h^\alpha_\mu - \partial_\mu \partial_\nu h\right) = -\Box h + \partial_{\mu\alpha} h^{\mu\alpha} \,.$$

The l.h.s. of the Einstein field equation $(*)$ is known as the Einstein tensor: $G_{\mu\nu} \stackrel{\text{def}}{=} R_{\mu\nu} - \dfrac{1}{2} R\, g_{\mu\nu}$.

(5) **f.** Show that, in linearized gravity,

$$G_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} + \partial_\mu \partial_\alpha h^\alpha_\nu + \partial_\alpha \partial_\nu h^\alpha_\mu - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \eta_{\mu\nu} \Box h\right) \,.$$

Again, in linearized gravity we may replace the product $R\, g_{\mu\nu} = R\,(\eta_{\mu\nu} + h_{\mu\nu})$ simply by $R\,\eta_{\mu\nu}$, since the Ricci scalar R is itself $\mathcal{O}(h)$. Given this, one straightforwardly substitutes the results obtained in d and e into the definition of the Einstein tensor to obtain the desired result.

Another common notational convention relies on the so-called trace-reversed metric perturbation $\overline{h}_{\mu\nu}$:

$$\overline{h}_{\mu\nu} \stackrel{\text{def}}{=} h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \,.$$

(5) **g.** Explain the attribute "trace-reversed".

Contraction with $\eta^{\mu\nu}$ readily reveals that $\overline{h} = \eta^{\mu\nu}\overline{h}_{\mu\nu} = \eta^{\mu\nu}(h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}) = h - 2h = -h$.

(5) **h.** Show that the definition of $\overline{h}_{\mu\nu}$ above is equivalent to $h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\overline{h}\eta_{\mu\nu}$.

We have, according to the definition, $h_{\mu\nu} = \overline{h}_{\mu\nu} + \frac{1}{2} h \eta_{\mu\nu}$. Substituting the result of g, i.e. $h = -\overline{h}$, on the r.h.s. yields the desired result.

(5) **i.** Show that, in terms of the trace-reversed metric perturbation, the linearized Einstein field equation is given by the linear partial differential equation

$$-\Box \overline{h}_{\mu\nu} + \partial_\mu \partial_\alpha \overline{h}^\alpha_\nu + \partial_\alpha \partial_\nu \overline{h}^\alpha_\mu - \eta_{\mu\nu} \partial_\alpha \partial_\beta \overline{h}^{\alpha\beta} = 16\pi T_{\mu\nu} \,.$$

The observation in h, viz. that $h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\overline{h}\eta_{\mu\nu}$, can be substituted directly into the Einstein tensor obtained in f, leading to some cancellations. Using $(*)$ then completes the proof.

THE END