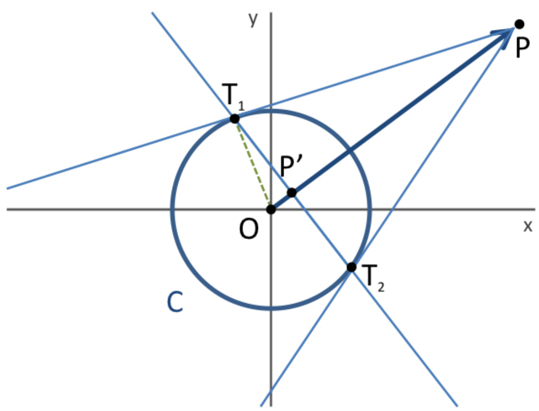
EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0-2WAH1. Date: Tuesday June 21, 2016. Time: 18h00–21h00. Place: AUD 11.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers.
- Einstein summation convention applies throughout for all repeated indices.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No equipment may be used.



INDICATRIX $C : x^2 + y^2 = 1$. THE POINT P′ IS THE INVERSION OF THE POINT P IN THE CIRCLE C.

(**30**) **1.** INDICATRIX

The so-called indicatrix C : $x^2 + y^2 = 1$ is a geometrical construction of the Euclidean metric (standard inner product) in the Cartesian coordinate plane. Assume point P $= (p, q)$ is given, somewhere outside C. There are two lines through P tangent to C, touching at the points T_1, respectively T_2. The so-called polar line T_1T_2 intersects the line OP at point P′ $= (m, n)$, say.

(5) **a1.** Show that OP : $1 = 1$: OP′.
[HINT: COMPARE THE TRIANGLES P′TO AND TOP, WITH T EITHER T_1 OR T_2.]

(5) **a2.** Show that the coordinates of P′ are given by $(m, n) = (\dfrac{p}{p^2 + q^2}, \dfrac{q}{p^2 + q^2})$.

(5) **b.** Argue why $\overrightarrow{\mathrm{OP}} \perp \mathrm{T}_1\mathrm{T}_2$, and show that the line $\mathrm{T}_1\mathrm{T}_2$ is given by $px + qy = 1$.

In summary, if $(m^1, m^2) = (m, n)$ and $(p^1, p^2) = (p, q)$ are the coordinates of $\overrightarrow{\mathrm{OP}'}$, respectively $\overrightarrow{\mathrm{OP}}$, $(x^1, x^2) = (x, y)$ any point on the indicatrix $\mathrm{C} : x^2 + y^2 = 1$, and η_{ij}, $i, j = 1, 2$, the matrix elements of the 2×2-identity matrix I, then, with $\cdot$ denoting the standard inner product on $\mathbb{R}^2$, we have

$$(\star) \qquad \left\{ \begin{array}{rrcl} \overrightarrow{\mathrm{OP}'} \cdot \overrightarrow{\mathrm{OP}} : & mp + nq & = & 1\,, \\ \mathrm{T}_1\mathrm{T}_2 : & px + qy & = & 1\,, \\ \mathrm{C} : & x^2 + y^2 & = & 1\,. \end{array} \right. \qquad \Longleftrightarrow \qquad \left\{ \begin{array}{rrcccl} \overrightarrow{\mathrm{OP}'} \cdot \overrightarrow{\mathrm{OP}} : & \eta_{ij} m^i p^j & = & m_j p^j & = & 1\,, \\ \mathrm{T}_1\mathrm{T}_2 : & \eta_{ij} p^i x^j & = & p_j x^j & = & 1\,, \\ \mathrm{C} : & \eta_{ij} x^i x^j & = & x_j x^j & = & 1\,. \end{array} \right.$$

The indicatrix C is a pictorial representation of the covariant Euclidean metric tensor $H = \eta_{ij} dx^i \otimes dx^j$. The polar line $\mathrm{T}_1\mathrm{T}_2$ provides a pictorial representation of the covector $\widehat{p} = p_i dx^i$ associated with the vector $\vec{p} = p^i \partial_i$ in the sense that it is just the differential of the left hand side of its defining equation. Note that the components p_i of the normal of the polar line happen to coincide numerically with the vector components p^i of $\vec{p} = \overrightarrow{\mathrm{OP}}$. This is an artefact of our choice of coordinates, or, alternatively, of the isotropic nature of C expressed in these coordinates, with $\eta_{ij} = 1$ if $i = j$ and zero otherwise.

Consider a linear change of coordinates, $x^i = L^i_j \overline{x}^j$, say, in which L^i_j, $i, j = 1, 2$, are the matrix entries of the transformation matrix

$$L = \begin{pmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

(10) **c.** Rewrite the system $(\star)$ for $\overrightarrow{\mathrm{OP}'} \cdot \overrightarrow{\mathrm{OP}}$, $\mathrm{T}_1\mathrm{T}_2$ and C, in terms of the new coordinates $(\overline{x}^1, \overline{x}^2) = (\overline{x}, \overline{y})$, the parameters m, n, p, q, and the components a, b, c, d of the transformation matrix L.

(5) **d.** Suppose we would redraw the aforementioned geometric construct (point P, tangent cords PT_1 and PT_2, polar line $\mathrm{T}_1\mathrm{T}_2$, spherical reflection P′, and indicatrix C, with all objects defined in terms of their $(\overline{x}, \overline{y})$-coordinatizations, in a *rectangular* $(\overline{x}, \overline{y})$-coordinate system in which basis (co)vectors are depicted as horizontal/vertical unit $\mathbb{R}^2$-coordinate (co)vectors:

$$\partial_{\overline{x}} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \partial_{\overline{y}} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Would the polar line $\mathrm{T}_1\mathrm{T}_2$ still be perpendicular to the vector $\overrightarrow{\mathrm{OP}}$? Motivate your answer by an explicit computation of the coordinates $\overline{p}^i$ and $\overline{p}_i$.

(**25**) **2.** KILLING VECTORS

The Lie derivative w.r.t. a given vector field $\mathbf{v}$ is a linear first order partial differential operator satisfying the product rule, such that

$$\mathscr{L}_{\mathbf{v}} f = \mathbf{v} f \quad \text{and} \quad \mathscr{L}_{\mathbf{v}} \hat{\boldsymbol{\omega}} = d\left(\hat{\boldsymbol{\omega}}(\mathbf{v})\right),$$

for a scalar field f, respectively a covector field $\hat{\boldsymbol{\omega}}$.

(5) **a.** Show that $\mathscr{L}_{\mathbf{v}} dx^i = \partial_k v^i dx^k$.

Let $g = g_{ij} dx^i \otimes dx^j$ denote the covariant metric tensor field. By D_i we denote the covariant derivative operator w.r.t. x^i induced by the Levi-Civita connection.

(5) **b1.** Show that $\mathscr{L}_{\mathbf{v}} g = (v^k \partial_k g_{ij} + \partial_i v^k g_{kj} + \partial_j v^k g_{ik})\, dx^i \otimes dx^j$.

(5) **b2.** Show that, equivalently, $\mathscr{L}_{\mathbf{v}} g = (D_i v_j + D_j v_i)\, dx^i \otimes dx^j$.

A vector field $\mathbf{v}$ is called a Killing vector field if $\mathscr{L}_{\mathbf{v}} g = 0$. We henceforth consider the Euclidean metric of flat 3-dimensional space in Cartesian coordinates $(x^1, x^2, x^3) = (x, y, z)$, and a Killing vector field with component functions $(v^1, v^2, v^3) = (u, v, w)$.

(10) **c.** Show that the Killing vector field must satisfy $u_x = v_y = w_z = 0$ and $v_x + u_y = w_x + u_z = w_y + v_z = 0$.

(**45**) **3.** LINEARIZED GRAVITY

In the presence of weak gravitational fields it is convenient to linearize the Einstein field equation for the gravitational potential, i.e. the pseudo-Riemannian spacetime metric tensor of General Relativity (GR):

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,.$$

Here, $\eta_{\mu\nu}$ is the holor of the Minkowski metric of Special Relativity (SR), in an "inertial" coordinate system given by the universally constant matrix $\eta_{\mu\nu} \widehat{=} \mathrm{diag}\,\{-1, 1, 1, 1\}$, with $\mu, \nu = 0, 1, 2, 3$. Index value 0 refers to the temporal basis vector ∂_t, index values $1, 2, 3$ to the three independent spatial basis vectors $\partial_x, \partial_y, \partial_z$.

In linearized gravity one thinks of the perturbation $h_{\mu\nu}$ as a spatiotemporal tensor field propagating on the constant Minkowski background spacetime. Its entries are spatiotemporal functions with values that are small relative to those of the background field: $|h_{\mu\nu}| \ll 1$. This allows one to approximate exact GR laws in terms of expansions to order $\mathcal{O}(h)$, neglecting higher order terms. This $\mathcal{O}(h)$ approximation is assumed throughout this problem without explicit indication of discarded higher order terms.

The (exact) Einstein's field equation is a tensorial nonlinear partial differential equation that governs the spacetime metric $g_{\mu\nu}$ in relation to the flux of energy and momentum, which is captured by the energy-momentum tensor field $T_{\mu\nu}$:

$$(*) \qquad R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi\, T_{\mu\nu}\,.$$

(Units are defined such that $G = c = 1$, with G: gravitational constant, c: velocity of light.) Recall that the Ricci tensor is given in terms of the Levi-Civita Christoffel symbols by

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\beta}_{\mu\alpha}\, \Gamma^{\alpha}_{\beta\nu} + \Gamma^{\beta}_{\mu\nu}\, \Gamma^{\alpha}_{\beta\alpha}\,.$$

An index preceded by a comma indicates a differentiation index: $f_{,\alpha} \stackrel{\text{def}}{=} \partial_\alpha f$.

(5) **a.** Show that $(*)$ reduces to $R_{\mu\nu} = 0$ in the homogeneous ("vacuum") case in which $T_{\mu\nu} = 0$.

Even this simplified vacuum field equation is a nonlinear partial differential equation for $g_{\mu\nu}$ that is not easily solved. Perturbation theory allows us to linearize $(*)$ and a fortiori its homogeneous counterpart.

(5) **b.** Show that, in linearized gravity, we may approximate $R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu}$, ignoring the $\Gamma\Gamma$-terms.

(5) **c.** Show that, in linearized gravity, $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\left(-\partial^\mu h_{\alpha\beta} + \partial_\beta h^{\mu}_{\alpha} + \partial_\alpha h^{\mu}_{\beta}\right)$. Explain why index raising in

this expression may be carried out with the help of the background (dual) metric $\eta^{\mu\nu}$ instead of $g^{\mu\nu}$.

(5) **d.** Show that, in linearized gravity, $R_{\mu\nu} = \frac{1}{2}\left(-\partial^\alpha\partial_\alpha h_{\mu\nu} + \partial_\mu\partial_\alpha h^\alpha_\nu + \partial_\alpha\partial_\nu h^\alpha_\mu - \partial_\mu\partial_\nu h^\alpha_\alpha\right)$.

One commonly abbreviates

$$h \stackrel{\text{def}}{=} h^\alpha_\alpha \quad \text{and} \quad \Box \stackrel{\text{def}}{=} \partial^\alpha\partial_\alpha \,,$$

so that

$$R_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} + \partial_\mu\partial_\alpha h^\alpha_\nu + \partial_\alpha\partial_\nu h^\alpha_\mu - \partial_\mu\partial_\nu h\right) \,.$$

Moreover, the Ricci scalar is defined as

$$R = R^\mu_\mu \,.$$

(5) **e.** Show that, in linearized gravity, $R = \partial_{\mu\alpha}h^{\mu\alpha} - \Box h$.

The l.h.s. of the Einstein field equation $(*)$ is known as the Einstein tensor: $G_{\mu\nu} \stackrel{\text{def}}{=} R_{\mu\nu} - \dfrac{1}{2} R\, g_{\mu\nu}$.

(5) **f.** Show that, in linearized gravity,

$$G_{\mu\nu} = \frac{1}{2}\Big(-\Box h_{\mu\nu} + \partial_\mu\partial_\alpha h^\alpha_\nu + \partial_\alpha\partial_\nu h^\alpha_\mu - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + \eta_{\mu\nu}\Box h\Big) \,.$$

Another common notational convention relies on the so-called trace-reversed metric perturbation $\overline{h}_{\mu\nu}$:

$$\overline{h}_{\mu\nu} \stackrel{\text{def}}{=} h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \,.$$

(5) **g.** Explain the attribute "trace-reversed".

(5) **h.** Show that the definition of $\overline{h}_{\mu\nu}$ above is equivalent to $h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\overline{h}\eta_{\mu\nu}$.

(5) **i.** Show that, in terms of the trace-reversed metric perturbation, the linearized Einstein field equation is given by the linear partial differential equation

$$-\Box\overline{h}_{\mu\nu} + \partial_\mu\partial_\alpha\overline{h}^\alpha_\nu + \partial_\alpha\partial_\nu\overline{h}^\alpha_\mu - \eta_{\mu\nu}\partial_\alpha\partial_\beta\overline{h}^{\alpha\beta} = 16\pi T_{\mu\nu} \,.$$

THE END