EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0-2WAH1. Date: Tuesday June 23, 2015. Time: 18h00–21h00. Place: AUD 14.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 2 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers in mathematically rigorous terms.
- Einstein summation convention applies throughout for all repeated pairs of upper and lower indices.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No equipment may be used.

(**50**) **1.** Riemannian Metric

Recall the positivity and non-degeneracy conditions for a Riemannian metric $\mathbf{G} : \mathrm{TM} \times \mathrm{TM} \to \mathbb{R}$: $\mathbf{G}(\mathbf{v}, \mathbf{v}) = (\mathbf{v}|\mathbf{v}) = g_{ij} v^i v^j \geq 0$ for any $\mathbf{v} = v^i \partial_i$ (positivity), with equality if and only if $\mathbf{v} = \mathbf{0}$ (non-degeneracy). We denote by g the determinant of the matrix with entries g_{ij}.

(5) **a1.** Show that, in general, positivity implies $g \geq 0$.

(5) **a2.** Show that, in general, non-degeneracy is equivalent to $g \neq 0$.

We now stipulate a particular 3-dimensional Riemannian metric of the form

$$\mathbf{G} = dx \otimes dx + dy \otimes dy + dz \otimes dz - (adx + bdy + cdz) \otimes (adx + bdy + cdz) \,,$$

in which $dx, dy, dz \in \mathrm{T}^*\mathrm{M}$ are coordinate one-forms and $a, b, c \in \mathbb{R}$ are constants.

(5) **b1.** Show that, in this case, non-degeneracy is equivalent to $a^2 + b^2 + c^2 \neq 1$.

(5) **b2.** Likewise, show that positive-definiteness of the metric implies $a^2 + b^2 + c^2 < 1$.

Next we consider the tensor $\mathbf{P} = p_{ij} dx^i \otimes dx^j$, with $p_{ij} = g_{ij} - \omega_i \omega_j$, in which now g_{ij} are the components of a genuine (positive-definite, non-degenerate) Riemannian metric, and $\omega_i = g_{ij} e^j$ are the covariant components of the dual $\boldsymbol{\omega} \in \mathrm{T}^*\mathrm{M}$ of a given *unit vector* $\mathbf{e} \in \mathrm{TM}$, i.e. $(\mathbf{e}|\mathbf{e}) = g_{ij} e^i e^j = 1$ and $\langle \boldsymbol{\omega}, \mathbf{e} \rangle = \omega_j e^j = 1$.

(5) **c.** Show that $\langle \mathbf{P}(\mathbf{u}), \mathbf{e}\rangle = 0$ for any vector $\mathbf{u} = u^i\partial_i$.

A special *g-orthonormal basis* $\{\mathbf{e}_\alpha = e^i_\alpha \partial_i\}_{\alpha=1,\ldots,n}$ relative to the metric $\mathbf{G} = g_{ij} dx^i \otimes dx^j$ is one for which $\mathbf{G}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = (\mathbf{e}_\alpha|\mathbf{e}_\beta) = g_{ij}e^i_\alpha e^j_\beta = \delta_{\alpha\beta}$, in which $\delta_{\alpha\beta} = 1$ and $\delta^{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise. The corresponding dual basis is $\{\boldsymbol{\omega}^\alpha = \omega^\alpha_i dx^i\}_{\alpha=1,\ldots,n}$.

(5) **d.** Show that g-orthonormality of a basis is preserved under orthogonal transformations.
(Hint: A matrix $\mathbf{A}$ corresponding to an orthogonal transformation satisfies $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.)

(5) **e1.** Show that $\delta^\alpha_\beta = \omega^\alpha_i e^i_\beta$.

(5) **e2.** Show that $\delta^i_j = e^i_\alpha \omega^\alpha_j$.
(Hint: Contract ω^β_j onto the result of e1.)

(5) **e3.** Show that $g_{ij} = \omega^\alpha_i \delta_{\alpha\beta}\omega^\beta_j = \sum_\alpha \omega^\alpha_i \omega^\alpha_j$.
(Hint: First show that e2 implies $\partial_i = \omega^\alpha_i \mathbf{e}_\alpha$.)

(5) **e4.** Show that $g^{ij} = e^i_\alpha \delta^{\alpha\beta} e^j_\beta = \sum_\alpha e^i_\alpha e^j_\alpha$.

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(**50**) **2.** MASSIVE VECTOR BOSONS

We consider 4-dimensional spacetime, endowed with a non-positive "Minkowski metric" $\boldsymbol{G} = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$. The corresponding non-positive, symmetric, non-degenerate bilinear form, or "inner product", of two 4-vectors, $\mathbf{v} = v^\mu \partial_\mu$ and $\mathbf{w} = w^\mu \partial_\mu$ say, is denoted by $\mathbf{v} \cdot \mathbf{w} = v^\mu \eta_{\mu\nu} w^\nu$.

We henceforth restrict ourselves to so-called "inertial systems", in which, by definition, the components of the covariant metric tensor are $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$, and $\eta_{\mu\nu} = 0$ otherwise ($\mu, \nu = 0, 1, 2, 3$). The defining inertial coordinates are indicated by $(x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \vec{x}) \in \mathbb{R}^4$, in which $c \approx 3.00 \times 10^8$ m/s denotes the universally constant speed of light in vacuum.

A so-called "Lorentz transformation" relates two inertial systems, with coordinates $x \in \mathbb{R}^4$ and $\overline{x} \in \mathbb{R}^4$, say, preserving the components of the Minkowski metric as stated above.

(5) **a1.** Show that if $x^\mu = L^\mu_\nu \overline{x}^\nu$ is a Lorentz transformation, then the components L^μ_ν must satisfy the condition $L^\mu_\rho \eta_{\mu\nu} L^\nu_\sigma = \eta_{\rho\sigma}$.
(Hint: Set $\overline{\eta}_{\mu\nu} d\overline{x}^\mu \otimes d\overline{x}^\nu = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$ and impose the defining *invariance* property $\overline{\eta}_{\mu\nu} = \eta_{\mu\nu}$.)

(5) **a2.** Show that Lorentz transformations are closed under composition. In other words, show that the effective transformation $x \to \overline{\overline{x}}$ resulting from two consecutive Lorentz transformations, $x \to \overline{x} \to \overline{\overline{x}}$ say, is itself a Lorentz transformation.

An *energy-momentum vector* $\mathbf{p} = p^\mu \partial_\mu$ has a corresponding representation in terms of a *wave vector* $\mathbf{k} = k^\mu \partial_\mu$ given by $\mathbf{p} = \hbar\mathbf{k}$, in which $\hbar \approx 1.05 \times 10^{-34}$ J s denotes Planck's constant. The following notation is often used for the components:

$$\mathbf{k} = (k^0, k^1, k^2, k^3) = (k^0, \vec{k}) = (\omega/c, \vec{k}) \,,$$

in which $\omega \in \mathbb{R}$ and $\vec{k} \in \mathbb{R}^3$ denote temporal angular frequency and spatial frequency 3-vector, and

$$\mathbf{p} = (p^0, p^1, p^2, p^3) = (p^0, \vec{p}) = (E/c, \vec{p}) \,,$$

in which $E \in \mathbb{R}$ and $\vec{p} \in \mathbb{R}^3$ denote energy and momentum 3-vector, respectively. These are to be distinguised from their dual (covector) counterparts $\widehat{\mathbf{k}} = k_\mu dx^\mu$ and $\widehat{\mathbf{p}} = p_\mu dx^\mu$.

($2\frac{1}{2}$) **b.** Show that, in any inertial frame, $\widehat{\mathbf{k}} = (k^0, -k^1, -k^2, -k^3) = (k^0, -\vec{k}) = (\omega/c, -\vec{k})$, respectively $\widehat{\mathbf{p}} = (p^0, -p^1, -p^2, -p^3) = (p^0, -\vec{p}) = (E/c, -\vec{p})$.

We consider a force field $F \stackrel{\text{def}}{=} dA$, generated by a covector-valued potential field $A = A_\nu dx^\nu$ through exterior derivation: $dA = \partial_\mu A_\nu \, dx^\mu \wedge dx^\nu$. The components of F relative to a standard tensor basis $\{dx^\mu \otimes dx^\nu\}$ are given by $F_{\mu\nu}$, i.e. $F = F_{\mu\nu} \, dx^\mu \otimes dx^\nu$.

($2\frac{1}{2}$) **c1.** Show that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

($2\frac{1}{2}$) **c2.** Show that, alternatively, $F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$.

The (contravariant components of the) potential field, $A^\mu = \eta^{\mu\nu} A_\nu$, satisfy the *Proca equation*,

$$\partial_\mu F^{\mu\nu} + \frac{m^2 c^2}{\hbar^2} A^\nu = 0 \,,$$

in which $m > 0$ is some mass parameter. This field equation describes so-called "massive vector bosons" (massive counterparts of the familiar massless photon in electromagnetism).

($2\frac{1}{2}$) **d1.** Show that the components of the potential field are not independent, but satisfy $\partial \cdot A = \partial_\mu A^\mu = 0$.

(5) **d2.** Show that the Proca equation can be rewritten as $(\partial^2 + \dfrac{m^2 c^2}{\hbar^2}) A^\nu = 0$, in which $\partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu$ (i.e. the so-called d'Alembertian).

We use the following Fourier convention in Minkowski space (in an inertial system):

$$f(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{-ik\cdot x} \hat{f}(k) dk \,,$$

in which $k \cdot x = \eta_{\mu\nu} k^\mu x^\nu$ and $dk = dk^0 dk^1 dk^2 dk^3$. (A $\hat{}$ henceforth denotes Fourier transform.) We take for granted that $f = 0$ is equivalent to $\hat{f} = 0$ ("completeness of the Fourier basis").

(5) **e1.** Show that the condition on the potential field (recall d1) is equivalent to $k \cdot \hat{A}(k) = \eta_{\mu\nu} k^\mu \hat{A}^\nu(k) = 0$.

(5) **e2.** Show that the Proca equation in Fourier space is $(-k^2 + \dfrac{m^2 c^2}{\hbar^2}) \hat{A}^\nu(k) = 0$, with $k^2 = k \cdot k = \eta^{\mu\nu} k_\mu k_\nu$.

For nontrivial Fourier amplitudes $\hat{A}^\nu(k)$ we apparently have the *dispersion relation* $-k^2 + \dfrac{m^2 c^2}{\hbar^2} = 0$.

(2½) **f1.** Show that this is just the Einstein's mass-energy equivalence relation $E^2 = m^2c^4 + \|\vec{p}\|^2c^2$ for a moving particle with mass m and 3-momentum $\vec{p}$.

Given the dispersion relation, the Proca equation admits plane wave solutions in Minkowski spacetime of the form $A^\mu(x;k) = \epsilon^\mu(k)e^{-ik\cdot x}$ parameterized by $k \in \mathbb{R}^4$. Note that $k \cdot \boldsymbol{\epsilon}(k) = \eta_{\mu\nu}k^\mu\epsilon^\nu(k) \stackrel{\text{e1}}{=} 0$.

(2½) **f2.** Show that there are precisely three independent "polarizations", i.e. independent components of the amplitude vector $\boldsymbol{\epsilon}(k) = \epsilon^\mu(k)\partial_\mu$ for each fixed $k \in \mathbb{R}^4$.

Thus admissible solutions are linear combinations of three "polarization states", which we shall denote by $\boldsymbol{\epsilon}_\ell(k) = \epsilon^\mu_\ell(k)\partial_\mu$, $\ell = 1, 2, 3$. Note that ℓ is a polarization state label, not a tensor index.

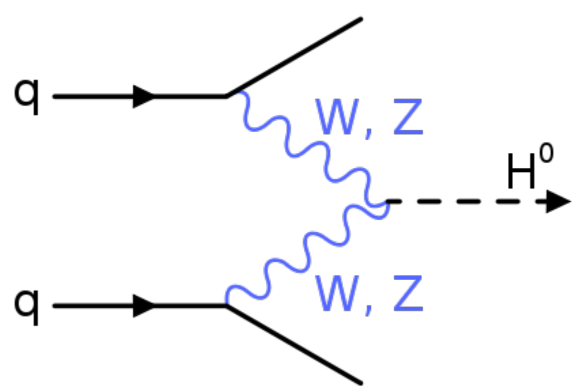
(2½) **g1.** Show that $\epsilon^0_\ell(k) = 0$ for all $\ell = 1, 2, 3$ if $\vec{k} = \vec{0}$.

(2½) **g2.** Show that, together with $\epsilon^0_\ell(k) = 0$, $\epsilon^i_\ell(k) = \delta^i_\ell$ for all $i, \ell = 1, 2, 3$ provides a desired set of three independent polarization states if $\vec{k} = \vec{0}$.

(5) **h.** Show that, for general wave vectors $k \in \mathbb{R}^4$ (i.e. without the constraint $\vec{k} = \vec{0}$),

$$\sum_{\ell=1}^{3} \epsilon^\mu_\ell(k)\epsilon^\nu_\ell(k) = \eta^{\mu\nu} - \frac{\hbar^2}{m^2c^2}k^\mu k^\nu .$$

(Hint: (i) The only contravariant 2-tensor holors that can be formed in this context are linear combinations of $\eta^{\mu\nu}$ and $k^\mu k^\nu$. (ii) Recall g1 and g2.)



FEYNMAN DIAGRAM: PRODUCTION OF A HIGGS BOSON (H^0) FROM MASSIVE VECTOR BOSONS (W, Z) EMITTED BY QUARKS (q).

THE END