EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0. Date: Tuesday June 26, 2018. Time: 18h00–21h00. Place: AUD 12

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No other material or equipment may be used.

(20) 1. Tensor Calculus Miscellany.

Consider the mixed tensor $\mathbf{X} = X^i_j\, \mathbf{e}_i \otimes \hat{\mathbf{e}}^j$, and a basis transformation $\mathbf{e}_i = A^j_i \mathbf{f}_j$, together with its induced dual basis transformation, $\hat{\mathbf{e}}^i = B^i_j \hat{\mathbf{f}}^j$. The transformed holor $\overline{X}^i_j$ is defined via $\mathbf{X} = \overline{X}^i_j\, \mathbf{f}_i \otimes \hat{\mathbf{f}}^j$.

(5) **a1.** Show that $A^k_i B^j_k = \delta^i_j$.

We have $\delta^j_i = \langle \hat{\mathbf{e}}^j, \mathbf{e}_i \rangle = \langle B^j_k \hat{\mathbf{f}}^k, A^\ell_i \mathbf{f}_\ell \rangle = A^\ell_i B^j_k \langle \hat{\mathbf{f}}^k, \hat{\mathbf{f}}_\ell \rangle = A^\ell_i B^j_k \delta^k_\ell = A^k_i B^j_k$.

(5) **a2.** Show that if the holor is invariant, i.e. if $\overline{X}^i_j = X^i_j$ for any basis transformation, then $\mathbf{X}$ must equal the Kronecker tensor up to a constant factor.

On the one hand we have the general 'tensor transformation law' $\overline{X}^i_j = A^i_k B^\ell_j X^k_\ell$. On the other hand we have the invariance requirement $\overline{X}^i_j = X^i_j$. Combining these, and using a, it follows that $A^j_k X^i_j = A^i_j X^j_k$ regardless of the choice of the transformation matrix. We may therefore regard the A^i_j as independent variables, and differentiate the equation with respect to these. This yields $X^i_j \delta^p_q = X^p_q \delta^i_j$. Contraction over $p = q$ produces $n X^i_j = X \delta^i_j$, in which $X = X^k_k$, in other words $X^i_j = c\delta^i_j$, in which $c = X/n$ is an arbitrary constant. We conclude that $\mathbf{X} = c\delta^i_j\, \mathbf{e}_i \otimes \hat{\mathbf{e}}^j = c\, \mathbf{e}_i \otimes \hat{\mathbf{e}}^i = c\, \langle\, \cdot\,, \cdot\, \rangle$ is a multiple of the Kronecker tensor.

Next, consider the tensors $\mathbf{V} = V^i \mathbf{e}_i$, $\mathbf{S} = S^{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathbf{T} = T_{ijk}\, \hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}^j \otimes \hat{\mathbf{e}}^k$.

(5) **b1.** Expand the holor $V^{(i}S^{jk)}$ of $\mathscr{S}(\mathbf{V} \otimes \mathbf{S})$, i.e. the symmetrised outer product of $\mathbf{V}$ and $\mathbf{S}$, explicitly in terms of the holors V^i and S^{jk}.

We have

$$V^{(i}S^{jk)} \doteq \mathscr{S}(\mathbf{V} \otimes \mathbf{S})^{ijk} = \frac{1}{6}\left(V^i S^{jk} + V^i S^{kj} + V^j S^{ik} + V^j S^{ki} + V^k S^{ij} + V^k S^{ji}\right).$$

(5) **b2.** Suppose $\mathscr{S}(\mathbf{V} \otimes \mathbf{S}) = 0$ for all vectors $\mathbf{V}$. Show that $\mathscr{S}(\mathbf{S}) = 0$.

Differentiating the null function

$$0 = f^{ijk}(\mathbf{V}) \doteq V^i S^{jk} + V^i S^{kj} + V^j S^{ik} + V^j S^{ki} + V^k S^{ij} + V^k S^{ji} = 2\left(V^i S^{(jk)} + V^j S^{(ik)} + V^k S^{(ij)}\right) \ ,$$

with respect to V^ℓ for fixed $\mathbf{S}$ yields

$$0 = \frac{1}{2}\frac{\partial f^{ijk}(\mathbf{V})}{\partial V^\ell} = \delta^i_\ell S^{(jk)} + \delta^j_\ell S^{(ik)} + \delta^k_\ell S^{(ij)} \ .$$

Contraction of the indices i and ℓ yields $(n+2)S^{(jk)} = 0$, in which n is the dimension of the vector space, whence $S^{(jk)} = 0$.

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(**30**) **2. LAPLACE-DE RHAM OPERATOR.**

The *Laplace-de Rham operator* Δ_{LDR}, a generalization of the *Laplace-Beltrami operator* Δ_{LB}, is defined so as to act on sections of any k-form bundle over a Riemannian manifold M of dimension n, $0 \leq k \leq n$. Its formal definition is

$$\Delta_{\text{LDR}} \doteq d\delta + \delta d\,,$$

in which d denotes the exterior derivative or differential, and δ the so-called *codifferential*. The latter is defined for a k-form field in terms of the differential d and Hodge star operator $*$ as follows:

$$\delta = (-1)^{(k+1)n+1} * d * \,.$$

For notational simplicity we denote the linear space of k-form fields by $\Lambda_k(\mathrm{M})$, $0 \leq k \leq n$. Recall that for a k-form field $\boldsymbol{\omega} = \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Lambda_k(\mathrm{M})$, say, the differential $d\boldsymbol{\omega} \in \Lambda_{k+1}(\mathrm{M})$ is defined as $d\boldsymbol{\omega} = \partial_i \omega_{i_1 \ldots i_k} dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, thus d increments covariant rank by one.

Terminology. A 0-form field is usually referred to as a *scalar field*, an n-form field as a *volume form*.

In this problem you may use the following lemma without proof.

Lemma. The double Hodge star $**$ preserves the covariant rank of its operand. More specifically, on a (positive definite) Riemannian manifold M of dimension n we have:

$$** = (-1)^{k(n-k)} 1_{\Lambda_k(\mathrm{M})}\,,$$

in which $1_{\Lambda_k(\mathrm{M})}$ is the identity operator (usually suppressed in the notation by writing $** = (-1)^{k(n-k)}$).

(5) **a1.** Show that for a scalar field $f \in \Lambda_0(\mathrm{M})$ we have $\Delta_{\text{LDR}} f = \delta d f$.

(5) **a2.** Show that for a volume form $\boldsymbol{\rho} \in \Lambda_n(\mathrm{M})$ we have $\Delta_{\text{LDR}} \boldsymbol{\rho} = d\delta\boldsymbol{\rho}$.

By definition $*1 = \boldsymbol{\epsilon} \in \Lambda_n(\mathrm{M})$, the unit n-form. Since the differential d increments covariant order of its operand, we have $d\boldsymbol{\epsilon} \in \Lambda_{n+1}(\mathrm{M})$, and therefore $d\boldsymbol{\epsilon} = 0$. By linearity we then also have $d * f = f d * 1 = f d\boldsymbol{\epsilon} = 0$, leading to the reduction $\Delta_{\text{LDR}} f = (d\delta + \delta d) f = \delta d f$. By the same token, $d\boldsymbol{\rho} \in \Lambda_{n+1}(\mathrm{M})$ and thus $d\boldsymbol{\rho} = 0$, whence $\Delta_{\text{LDR}} \boldsymbol{\rho} = (d\delta + \delta d)\boldsymbol{\rho} = d\delta\boldsymbol{\rho}$.

(5) **b.** Argue that if $\boldsymbol{\omega} \in \Lambda_k(\mathrm{M})$, then $\delta\boldsymbol{\omega} \in \Lambda_{k-1}(\mathrm{M})$, i.e. δ decrements covariant rank by one.
[*Hint:* Avoid explicit calculation of $\delta\boldsymbol{\omega}$.]

Use the prototypes of $* : \Lambda_k(\mathrm{M}) \to \Lambda_{n-k}(\mathrm{M})$ and $d : \Lambda_k(\mathrm{M}) \to \Lambda_{k+1}(\mathrm{M})$ for any $0 \leq k \leq n$. Ignoring the irrelevant sign, we thus have $\delta(k\text{-form}) = (*d*)(k\text{-form}) = (*d)((n-k)\text{-form}) = *((n-k+1)\text{-form}) = (n-(n-k+1))\text{-form} = (k-1)\text{-form}$.

(5) **c.** Show that we may also write the definition of the Laplace-de Rham operator as $\Delta_{\text{LDR}} = (d + \delta)^2$.

This boils down to proving that $d^2 = \delta^2 = 0$.

Proof of $d^2 = 0$: Let $\boldsymbol{\omega} \in \Lambda_k(\mathrm{M})$ be a section of the k-form bundle, say $\boldsymbol{\omega} = \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$. Then $d\boldsymbol{\omega} \in \Lambda_{k+1}(\mathrm{M})$ is defined as $d\boldsymbol{\omega} = \partial_i \omega_{i_1 \ldots i_k} dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, and $d^2\boldsymbol{\omega} = \partial_j \partial_i \omega_{i_1 \ldots i_k} dx^j \wedge dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} = 0$. The latter follows from symmetry considerations w.r.t. the (i,j)-sum: $\partial_j \partial_i$ is symmetric, whereas $dx^j \wedge dx^i$ antisymmetric.

Proof of $\delta^2 = 0$: $\delta^2 \propto (*d*)(*d*) \overset{\dagger}{\propto} *d^2* \overset{\ddagger}{=} 0$, in which $\propto$ indicates equality up to a constant (sign) factor. In the step marked by $\dagger$ we have used the fact that $** \propto 1$ is the identity operator up to a possible minus sign*. In the step marked by $\ddagger$ we have used $d^2 = 0$.

*As an aside, the exact form of the double Hodge star in a pseudo-Riemannian manifold with metric determinant $g \neq 0$ is given by $** = (-1)^{k(n-k)} \operatorname{sgn} g$. The codifferential then also acquires a possible minus sign: $\delta = (-1)^{(k+1)n+1} \operatorname{sgn} g * d *$.

For a scalar field f the Laplace-Beltrami operator Δ_{LB} is defined as

$$\Delta_{\text{LB}} f \doteq \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f \right) ,$$

with g the determinant of the Gram matrix with entries g_{ij}, and g^{ij} the holor of the inverse Gram matrix.

(10) **d.** Derive the relation between $\Delta_{\text{LDR}} f$ and $\Delta_{\text{LB}} f$ for a scalar field f.
[*Hint:* (i) $*(\omega_k dx^k) = g^{k\ell} \omega_k \epsilon_{\ell i_2 \ldots i_n} dx^{i_2} \otimes \ldots \otimes dx^{i_n}$; (ii) $** \, 1 = 1$; (iii) $\epsilon_{i_1 \ldots i_n} = \sqrt{g} \, [i_1, \ldots, i_n]$.]

We have $\Delta_{\text{LDR}} f = \delta df = - * d * df$. Notice the minus sign, which originates from the definition of δ acting on a 1-form (viz. df). Working this out from right to left we first obtain

$$df = \partial_k f dx^k .$$

Taking the Hodge star yields, using the hint,

$$*df = (g^{k\ell} \sqrt{g} \partial_k f) \mu_{\ell i_2 \ldots i_n} dx^{i_2} \otimes \ldots \otimes dx^{i_n} ,$$

in which the notation $\mu_{i_1 \ldots i_n} = [i_1, \ldots, i_n]$ has been used merely to maintain manifest covariant indices. The factor $\sqrt{g}$ has been factored out so as to clarify the x-dependence of the holor of this $(n-1)$-form, viz. the part in parentheses. Taking the differential yields

$$d * df = \partial_m \left(g^{k\ell} \sqrt{g} \partial_k f \right) \mu_{\ell i_2 \ldots i_n} dx^m \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_n} .$$

Putting back the original holor of the $\boldsymbol{\epsilon}$-tensor, $\sqrt{g} \, \mu_{\ell i_2 \ldots i_n} = \epsilon_{\ell i_2 \ldots i_n}$, we get

$$d * df = \frac{1}{\sqrt{g}} \partial_m \left(g^{k\ell} \sqrt{g} \partial_k f \right) \epsilon_{\ell i_2 \ldots i_n} dx^m \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_n} .$$

Consider the tensor $\epsilon_{\ell i_2 \ldots i_n} dx^m \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_n}$. Since all indices $i_2, \ldots, i_n \in \{1, \ldots, n\}$ are effectively distinct, the only nontrivial option for the free index m results from the choice $m = \ell$, distinct from each $i_2, \ldots, i_n$. As a result we have

$$\epsilon_{\ell i_2 \ldots i_n} dx^m \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_n} = \delta_\ell^m \epsilon_{i_1 \ldots i_n} dx^{i_1} \otimes \ldots \otimes dx^{i_n} = \delta_\ell^m \boldsymbol{\epsilon} .$$

Consequently,

$$d * df = \frac{1}{\sqrt{g}} \partial_\ell \left(g^{k\ell} \sqrt{g} \partial_k f \right) \boldsymbol{\epsilon} .$$

Next, taking the Hodge dual once again, using $*\boldsymbol{\epsilon} = ** \, 1 = 1$, we obtain

$$*d * df = \frac{1}{\sqrt{g}} \partial_\ell \left(g^{k\ell} \sqrt{g} \partial_k f \right) .$$

Finally, looking at the original definition, including a minus sign, we observe that

$$\Delta_{\text{LDR}} f \doteq - * d * df \doteq - \Delta_{\text{LB}} f .$$

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(**50**) **3. RIEMANN, RICCI & COTTON TENSOR.**

Recall the definition of the (holor of the) *Riemann tensor* on an n-dimensional Riemannian manifold in terms of the Levi-Civita Christoffel symbols Γ^k_{ij}:

$$R^k_{i\ell j} = \partial_\ell \Gamma^k_{ij} - \partial_j \Gamma^k_{i\ell} + \Gamma^k_{\lambda\ell}\Gamma^\lambda_{ij} - \Gamma^k_{\lambda j}\Gamma^\lambda_{i\ell}\,.$$

The (holor of the) *Ricci tensor* is defined as

$$R_{ij} = R^k_{ikj}\,. \qquad (*)$$

By definition, the fully covariant Riemann tensor has holor

$$R_{ijk\ell} = g_{im} R^m_{jk\ell}\,.$$

(5) **a.** Show that $R_{ij} = R_{ji}$.
[*Hint:* Write $R_{ij} = R^k_{ikj} = g^{k\ell} R_{\ell ikj}$ and observe that $R_{\ell ikj} = R_{kj\ell i}$.]

Following the hint we write

$$R_{ij} = R^k_{ikj} = g^{k\ell} R_{\ell ikj} = g^{k\ell} R_{kj\ell i} = R^\ell_{j\ell i} = R_{ji}\,.$$

The *Ricci scalar* is the trace of the Ricci tensor: $R = g^{ij} R_{ij}$. Henceforth, D_k refers to a covariant derivative corresponding to the Levi-Civita connection.

(5) **b.** Expand $D_k R_{ij}$ in terms of (0^{th} and 1^{st} order) partial derivatives of R_{ij} and Christoffel symbols, but do *not* expand R_{ij} itself.

$D_k R_{ij} = \partial_k R_{ij} - \Gamma^\ell_{ik} R_{\ell j} - \Gamma^\ell_{jk} R_{i\ell}$.

(5) **c.** Show that, in dimension $n=1$, $R_{ijk\ell} = 0$.
[*Hint:* The covariant holor $R_{ijk\ell}$ inherits a certain antisymmetry property from the mixed holor $R^m_{jk\ell}$.]

By antisymmetry $R_{ijk\ell} = -R_{ij\ell k}$ we have, in $n=1$, $R_{1111} = -R_{1111}$ as the single degree of freedom, whence $R_{ijk\ell} = 0$.

Without proof we state that the fully covariant Riemann tensor in $n=2$ has the following form:

$$n = 2: \qquad R_{ijk\ell} = (g_{ik} g_{j\ell} - g_{i\ell} g_{jk})\,\psi\,,$$

for some scalar field $\psi \in C^\infty(\mathrm{M})$.

(5) **d.** Show that, in this 2-dimensional case, $\psi = \frac{1}{2} R$.

We have $R_{ij} = g^{k\ell} R_{\ell ikj} \stackrel{\star}{=} g^{k\ell}(g_{\ell k} g_{ij} - g_{\ell j} g_{ik})\,\psi = (n g_{ij} - g_{ij})\psi \stackrel{n=2}{=} g_{ij}\psi$, whence $R \stackrel{\text{def}}{=} g_{ij} R_{ij} = g^{ij} g_{ij} \psi = n\psi \stackrel{\star}{=} 2\psi$. The identities marked with $\star$ exploit the conjecture for $n=2$.

Likewise without proof we state that, in $n=3$ dimensions,

$$n = 3: \qquad R_{ijk\ell} = a(g_{ik} R_{j\ell} - g_{jk} R_{i\ell} - g_{i\ell} R_{jk} + g_{j\ell} R_{ik}) + b(g_{ik} g_{j\ell} - g_{i\ell} g_{jk}) R\,,$$

for certain constants $a, b \in \mathbb{R}$.

(5) **e.** Show that $a = 1$ and $b = -\frac{1}{2}$.

Contraction with g^{ik} yields $R_{j\ell} \stackrel{\text{def}}{=} g^{ik} R_{ijk\ell} \stackrel{\star}{=} g^{ik} \left(a(g_{ik}R_{j\ell} - g_{jk}R_{i\ell} - g_{i\ell}R_{jk} + g_{j\ell}R_{ik}) + b(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) \right)$. The identity marked with $\star$ exploits the conjecture for $n=3$. Simplifying the r.h.s. we obtain, using $g^{ik}R_{ik} = R$ and $g^{ik}g_{ik} = n = 3$ in this specific case: $R_{j\ell} = (n-2)aR_{j\ell} + (a + (n-1)b)g_{j\ell}R \stackrel{n=3}{=} aR_{j\ell} + (a + 2b)g_{j\ell}R$, whence $a = 1$ and $b = -\frac{1}{2}$.

We consider a conformal metric transformation, $\widetilde{g}_{ij} = e^{2\phi}g_{ij}$, in which $\phi \in C^\infty(\mathrm{M})$ is a smooth scalar field. The Levi-Civita Christoffel symbols associated with the transformed metric are indicated by $\widetilde{\Gamma}^k_{ij}$.

(5) **f1.** Show that $\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + S^k_{ij}$ and derive the explicit form of the symbols S^k_{ij} in terms of the scalar field ϕ and the original Riemannian metric tensor g_{ij}.

We have $S^k_{ij} = \partial_i\phi\delta^k_j + \partial_j\phi\delta^k_i - \partial^k\phi g_{ij}$, in which $\partial^k\phi = g^{k\ell}\partial_\ell\phi$.

(5) **f2.** Argue why S^k_{ij}, as opposed to Γ^k_{ij}, is the holor of a tensor.
[*Hint:* Avoid elaborate computations. Use the observations $D_k g_{ij} = 0$ and $D_k f = \partial_k f$ for any scalar field f.]

In the expression $S^k_{ij} = \partial_i\phi\delta^k_j + \partial_j\phi\delta^k_i - \partial^k\phi g_{ij}$ we may, following the hint, replace each ∂ by D, showing the tensorial nature of S^k_{ij}.

(5) **g1.** Show that in general dimension n the holor $\widetilde{R}^k_{i\ell j}$ of the Riemann tensor after conformal metric transformation is given in terms of $R^k_{i\ell j}$ and the tensorial holor S^k_{ij} as follows:

$$\widetilde{R}^k_{i\ell j} = R^k_{i\ell j} + D_\ell S^k_{ij} - D_j S^k_{i\ell} + S^k_{\lambda\ell}S^\lambda_{ij} - S^k_{\lambda j}S^\lambda_{i\ell}\,.$$

Insert $\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + S^k_{ij}$ into definition of $\widetilde{R}^k_{i\ell j}$ in terms of $\widetilde{\Gamma}$-symbols, and the result follows.

(5) **g2.** Now take $n=2$. Find $\widetilde{R}_{ij}$ in terms of R_{ij} and ϕ.

Using the definition of the Ricci tensor and its simplification in $n=2$, $R_{ij} = g^{k\ell}R_{\ell ikj} \stackrel{\text{d}}{=} \frac{1}{2}Rg_{ij}$, with Γ^k_{ij} replaced by $\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + S^k_{ij}$, we obtain, using f1, $\widetilde{R}_{ij} = R_{ij} - g_{ij}\Delta_{\text{LB}}\phi$, in which $\Delta_{\text{LB}}\phi = D^kD_k\phi = D^k\partial_k\phi$.

Note: In general dimension n we have $\widetilde{R}_{ij} = R_{ij} - g_{ij}\Delta_{\text{LB}}\phi - (n-2)\left(\partial_i\phi\partial_j\phi - D_i\partial_j\phi - \|\nabla\phi\|^2 g_{ij}\right)$. This general result follows from f1 and g1 after a more tedious but straightforward computation.

The so-called *Cotton tensor* is a third order covariant tensor with the following holor:

$$C_{ijk} = D_kR_{ij} - D_jR_{ik} + \frac{1}{2(n-1)}\left(g_{ik}D_jR - g_{ij}D_kR\right)\,.$$

(5) **h.** Show that in $n=2$ the Cotton vanishes identically: $C_{ijk} = 0$.

Using the $n = 2$ form of the fully covariant Riemann tensor we find $C_{ijk} = \dfrac{1}{2}(g_{jk}D_jR - g_{ij}D_kR)$. With the help of e we obtain $R_{ij} = \frac{1}{2}Rg_{ij}$. Inserting this in the definition of the Cotton tensor yields (remember to take $n=2$) $C_{ijk} = 0$.

THE END