EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0. Date: Tuesday June 26, 2018. Time: 18h00–21h00. Place: AUD 12

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No other material or equipment may be used.

(**20**) **1. TENSOR CALCULUS MISCELLANY.**

Consider the mixed tensor $\mathbf{X} = X^i_j\, \mathbf{e}_i \otimes \hat{\mathbf{e}}^j$, and a basis transformation $\mathbf{e}_i = A^j_i \mathbf{f}_j$, together with its induced dual basis transformation, $\hat{\mathbf{e}}^i = B^i_j \hat{\mathbf{f}}^j$. The transformed holor $\overline{X}^i_j$ is defined via $\mathbf{X} = \overline{X}^i_j\, \mathbf{f}_i \otimes \hat{\mathbf{f}}^j$.

(5) **a1.** Show that $A^k_i B^j_k = \delta^i_j$.

(5) **a2.** Show that if the holor is invariant, i.e. if $\overline{X}^i_j = X^i_j$ for any basis transformation, then $\mathbf{X}$ must equal the Kronecker tensor up to a constant factor.

Next, consider the tensors $\mathbf{V} = V^i \mathbf{e}_i$, $\mathbf{S} = S^{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathbf{T} = T_{ijk}\, \hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}^j \otimes \hat{\mathbf{e}}^k$.

(5) **b1.** Expand the holor $V^{(i}S^{jk)}$ of $\mathscr{S}(\mathbf{V} \otimes \mathbf{S})$, i.e. the symmetrised outer product of $\mathbf{V}$ and $\mathbf{S}$, explicitly in terms of the holors V^i and S^{jk}.

(5) **b2.** Suppose $\mathscr{S}(\mathbf{V} \otimes \mathbf{S}) = 0$ for all vectors $\mathbf{V}$. Show that $\mathscr{S}(\mathbf{S}) = 0$.

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(**30**) **2. LAPLACE-DE RHAM OPERATOR.**

The *Laplace-de Rham operator* Δ_{LDR}, a generalization of the *Laplace-Beltrami operator* Δ_{LB}, is defined so as to act on sections of any k-form bundle over a Riemannian manifold M of dimension n, $0 \leq k \leq n$. Its formal definition is

$$\Delta_{\text{LDR}} \doteq d\delta + \delta d\,,$$

in which d denotes the exterior derivative or differential, and δ the so-called *codifferential*. The latter is defined for a k-form field in terms of the differential d and Hodge star operator $*$ as follows:

$$\delta = (-1)^{(k+1)n+1} * d * \,.$$

For notational simplicity we denote the linear space of k-form fields by $\Lambda_k(\mathrm{M})$, $0 \leq k \leq n$.

Terminology. A 0-form field is usually referred to as a *scalar field*, an n-form field as a *volume form*.

In this problem you may use the following lemma without proof.

Lemma. The double Hodge star $**$ preserves the covariant rank of its operand. More specifically, on a (positive definite) Riemannian manifold M of dimension n we have:

$$** = (-1)^{k(n-k)} 1_{\Lambda_k(\mathrm{M})}\,,$$

in which $1_{\Lambda_k(\mathrm{M})}$ is the identity operator (often simply written as 1, or even suppressed altogether).

(5) **a1.** Show that for a scalar field $f \in \Lambda_0(\mathrm{M})$ we have $\Delta_{\text{LDR}} f = \delta d f$.

(5) **a2.** Show that for a volume form $\boldsymbol{\rho} \in \Lambda_n(\mathrm{M})$ we have $\Delta_{\text{LDR}} \boldsymbol{\rho} = d\delta\boldsymbol{\rho}$.

Recall that for a k-form field $\boldsymbol{\omega} = \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Lambda_k(\mathrm{M})$, say, the differential $d\boldsymbol{\omega} \in \Lambda_{k+1}(\mathrm{M})$ is defined as $d\boldsymbol{\omega} = \partial_i \omega_{i_1 \ldots i_k} dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, thus d increments covariant rank by one.

(5) **b.** Argue that if $\boldsymbol{\omega} \in \Lambda_k(\mathrm{M})$, then $\delta\boldsymbol{\omega} \in \Lambda_{k-1}(\mathrm{M})$, i.e. δ decrements covariant rank by one.
[*Hint:* Avoid explicit calculation of $\delta\boldsymbol{\omega}$.]

(5) **c.** Show that we may also write the definition of the Laplace-de Rham operator as $\Delta_{\text{LDR}} = (d + \delta)^2$.

For a scalar field f the Laplace-Beltrami operator Δ_{LB} is defined as

$$\Delta_{\text{LB}} f \doteq \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f \right)\,,$$

with g the determinant of the Gram matrix with entries g_{ij}, and g^{ij} the holor of the inverse Gram matrix.

(10) **d.** Derive the relation between $\Delta_{\text{LDR}} f$ and $\Delta_{\text{LB}} f$ for a scalar field f.
[*Hint:* You may use: (i) $*(\omega_k dx^k) = g^{k\ell} \omega_k \epsilon_{\ell i_2 \ldots i_n} dx^{i_2} \otimes \ldots \otimes dx^{i_n}$, (ii) $**1 = 1$, (iii) $\epsilon_{i_1 \ldots i_n} = \sqrt{g}\,[i_1, \ldots, i_n]$.]

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(50) **3. Riemann, Ricci & Cotton Tensor.**

Recall the definition of the (holor of the) *Riemann tensor* on an n-dimensional Riemannian manifold in terms of the Levi-Civita Christoffel symbols Γ^k_{ij}:

$$R^k_{i\ell j} = \partial_\ell \Gamma^k_{ij} - \partial_j \Gamma^k_{i\ell} + \Gamma^k_{\lambda\ell}\Gamma^\lambda_{ij} - \Gamma^k_{\lambda j}\Gamma^\lambda_{i\ell}\,.$$

The (holor of the) *Ricci tensor* is defined as

$$R_{ij} = R^k_{ikj}\,. \qquad (*)$$

By definition, the fully covariant Riemann tensor has holor

$$R_{ijk\ell} = g_{im} R^m_{jk\ell}\,.$$

(5) **a.** Show that $R_{ij} = R_{ji}$.
[*Hint:* Write $R_{ij} = R^k_{ikj} = g^{k\ell} R_{\ell ikj}$ and observe that $R_{\ell ikj} = R_{kj\ell i}$.]

The *Ricci scalar* is the trace of the Ricci tensor: $R = g^{ij} R_{ij}$. Henceforth, D_k refers to a covariant derivative corresponding to the Levi-Civita connection.

(5) **b.** Expand $D_k R_{ij}$ in terms of (0$^{\text{th}}$ and 1$^{\text{st}}$ order) partial derivatives of R_{ij} and Christoffel symbols, but do *not* expand R_{ij} itself.

(5) **c.** Show that, in dimension $n=1$, $R_{ijk\ell} = 0$.
[*Hint:* Recall the symmetry property in the previous hint.]

Without proof we state that the fully covariant Riemann tensor in $n=2$ has the following form:

$$n = 2: \qquad R_{ijk\ell} = (g_{ik} g_{j\ell} - g_{i\ell} g_{jk})\,\psi\,,$$

for some scalar field $\psi \in C^\infty(\mathrm{M})$.

(5) **d.** Show that, in this 2-dimensional case, $\psi = \frac{1}{2}R$.

Likewise without proof we state that, in $n=3$ dimensions,

$$n = 3: \qquad R_{ijk\ell} = a(g_{ik}R_{j\ell} - g_{jk}R_{i\ell} - g_{i\ell}R_{jk} + g_{j\ell}R_{ik}) + b(g_{ik}g_{j\ell} - g_{i\ell}g_{jk})R\,,$$

for certain constants $a, b \in \mathbb{R}$.

(5) **e.** Show that $a = 1$ and $b = -\frac{1}{2}$.

We consider a conformal metric transformation, $\widetilde{g}_{ij} = e^{2\phi} g_{ij}$, in which $\phi \in C^\infty(\mathrm{M})$ is a smooth scalar field. The Levi-Civita Christoffel symbols associated with the transformed metric are indicated by $\widetilde{\Gamma}^k_{ij}$.

(5) **f1.** Show that $\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + S^k_{ij}$ and derive the explicit form of the symbols S^k_{ij} in terms of the scalar field ϕ and the original Riemannian metric tensor g_{ij}.

(5) **f2.** Argue why S^k_{ij}, as opposed to Γ^k_{ij}, is the holor of a tensor.
[*Hint:* Avoid elaborate computations. Use the observations $D_k g_{ij} = 0$ and $D_k f = \partial_k f$ for any scalar field f.]

(5) **g1.** Show that in general dimension n the holor $\widetilde{R}^k_{i\ell j}$ of the Riemann tensor after conformal metric transformation is given in terms of $R^k_{i\ell j}$ and the tensorial holor S^k_{ij} as follows:

$$\widetilde{R}^k_{i\ell j} = R^k_{i\ell j} + D_\ell S^k_{ij} - D_j S^k_{i\ell} + S^k_{\lambda\ell} S^\lambda_{ij} - S^k_{\lambda j} S^\lambda_{i\ell} \,.$$

(5) **g2.** Now take $n=2$. Find $\widetilde{R}_{ij}$ in terms of R_{ij} and ϕ.

The so-called *Cotton tensor* is a third order covariant tensor with the following holor:

$$C_{ijk} = D_k R_{ij} - D_j R_{ik} + \frac{1}{2(n-1)} \left(g_{ik} D_j R - g_{ij} D_k R \right) \,.$$

(5) **h.** Show that in $n=2$ the Cotton vanishes identically: $C_{ijk} = 0$.

THE END