EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0. Date: Thursday June 27, 2017. Time: 18h00–21h00. Place: AUD 15.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- Einstein summation convention applies throughout for all repeated indices.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No equipment may be used.

(**20**) **1. STRAIN IN EUCLIDEAN SPACE.**

Definition. The Lie derivative of a smooth scalar field $f \in C^\infty(\mathbb{R}^n)$ with respect to a smooth vector field $\mathbf{v} \in \mathrm{T}\mathbb{R}^n$ is given by its directional derivative

$$\mathscr{L}_{\mathbf{v}} f = \mathbf{v} f \,,$$

with $\mathbf{v} f = v^k \partial_k f$.

Definition. The Lie derivative of a smooth covector field $\boldsymbol{\omega} \in \mathrm{T}^*\mathbb{R}^n$ with respect to a smooth vector field $\mathbf{v} \in \mathrm{T}\mathbb{R}^n$ is defined as

$$\mathscr{L}_{\mathbf{v}} \boldsymbol{\omega} = d\langle \boldsymbol{\omega}, \mathbf{v} \rangle \,,$$

in which $\langle \, , \, \rangle \in \mathrm{T}^*\mathbb{R}^n \otimes \mathrm{T}\mathbb{R}^n$ is the Kronecker tensor field and d denotes the differential with respect to the (implicit) basepoint coordinate chart $x \in \mathbb{R}^n$, i.e. $df = \partial_k f dx^k$ for any $f \in C^\infty(\mathbb{R}^n)$.

Definition. The Lie derivative is extended to general cotensors $\mathbf{T} \in \mathrm{T}^*\mathbb{R}^n \otimes \ldots \otimes \mathrm{T}^*\mathbb{R}^n$ via linearity and product rule in the usual way. For example, if $\mathbf{g} = g_{ij} \, dx^i \otimes dx^j \in \mathrm{T}^*\mathbb{R}^n \otimes \mathrm{T}^*\mathbb{R}^n$ with smooth coefficient functions $g_{ij} \in C^\infty(\mathbb{R}^n)$, then $\mathscr{L}_{\mathbf{v}} \mathbf{g} = (\mathscr{L}_{\mathbf{v}} g_{ij}) \, dx^i \otimes dx^j + g_{ij} \, (\mathscr{L}_{\mathbf{v}} dx^i) \otimes dx^j + g_{ij} \, dx^i \otimes (\mathscr{L}_{\mathbf{v}} dx^j)$.

Terminology. If $\mathbf{g}$ is the metric tensor field, with holor $g_{ij} = (\partial_i | \partial_j)$, then $\mathscr{L}_{\mathbf{v}} \mathbf{g}$ is known as the *strain tensor field* with respect to the displacement field $\mathbf{v}$ induced by a deformation of the medium.

(10) **a.** Compute the holor of the strain tensor field $\mathscr{L}_{\mathbf{v}} \mathbf{g}$ in a general coordinate chart.

We have $\mathscr{L}_{\mathbf{v}} \mathbf{g} = \mathscr{L}_{\mathbf{v}} (g_{ij} \, dx^i \otimes dx^j) = (\mathscr{L}_{\mathbf{v}} g_{ij}) \, dx^i \otimes dx^j + g_{ij} \, (\mathscr{L}_{\mathbf{v}} dx^i) \otimes dx^j + g_{ij} \, dx^i \otimes (\mathscr{L}_{\mathbf{v}} dx^j)$, i.e., following the definitions, $\mathscr{L}_{\mathbf{v}} \mathbf{g} = \mathbf{v} g_{ij} \, dx^i \otimes dx^j + g_{ij} \, d\langle dx^i, \mathbf{v} \rangle \otimes dx^j + g_{ij} \, dx^i \otimes d\langle dx^j, \mathbf{v} \rangle = v^k \partial_k g_{ij} \, dx^i \otimes dx^j + g_{ij} \, \partial_k v^i dx^k \otimes dx^j + g_{ij} \, dx^i \otimes \partial_k v^j dx^k$. Rearranging dummies this can be simplified to $\mathscr{L}_{\mathbf{v}} \mathbf{g} = (v^k \partial_k g_{ij} + \partial_i v^k g_{kj} + \partial_j v^k g_{ik}) \, dx^i \otimes dx^j$, in which one recognizes the holor.

(10) **b.** Show that in Cartesian coordinates the strain tensor field simplifies to $\mathscr{L}_{\mathbf{v}}\mathbf{g} = (\partial_i v_j + \partial_j v_i)\, dx^i \otimes dx^j$.

Using the previous result in combination with $g_{ij} = 1$ iff $i = j$ and zero otherwise, and denoting $v_i = g_{ij} v^j$ as usual, the result follows straightforwardly by virtue of global constancy of g_{ij}.

♣

(40) 2. Hodge Star and Faraday 2-Form in Minkowski Spacetime.

Definition. We consider 4-dimensional Minkowski spacetime M, furnished with a Lorentzian metric $(\mathbf{x}|\mathbf{y}) = G(\mathbf{x}, \mathbf{y}) = g_{ij} x^i y^j$. The components of the corresponding Gram matrix $\mathbf{G}$ are $g_{ij} = (\partial_i|\partial_j)$ $(i, j = 1, 2, 3, 4)$. We abbreviate $g = \det \mathbf{G}$.

Definition. We employ canonical coordinates, in terms of which $g_{11} = g_{22} = g_{33} = 1$, $g_{44} = -1$, $g_{ij} = 0$ otherwise. We abbreviate $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\partial_3 = \partial_z$, $\partial_4 = \partial_t$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$, $dx^4 = dt$.

Definition. The Levi-Civita tensor in n dimensions is given by

$$\boldsymbol{\epsilon} = \sqrt{|g|}\, dx^1 \wedge \ldots \wedge dx^n = \epsilon_{|i_1 \ldots i_n|} dx^{i_1} \wedge \ldots \wedge dx^{i_n} .$$

Definition. The linear Hodge star operator $* : \mathrm{T}^*\mathrm{M} \underbrace{\otimes_A \ldots \otimes_A}_{k} \mathrm{T}^*\mathrm{M} \to \mathrm{T}^*\mathrm{M} \underbrace{\otimes_A \ldots \otimes_A}_{n-k} \mathrm{T}^*\mathrm{M}$ satisfies

$$*\left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) = \frac{1}{(n-k)!} g^{i_1 j_1} \ldots g^{i_k j_k} \epsilon_{j_1 \ldots j_n} dx^{j_{k+1}} \wedge \ldots \wedge dx^{j_n} .$$

(20) **a.** Compute the following forms in terms of wedge products of dx, dy, dz, and dt:

a1. $*1$.

a2. $*dx$; $*dy$; $*dz$; $*dt$.

a3. $*(dx \wedge dy)$; $*(dx \wedge dz)$; $*(dx \wedge dt)$; $*(dy \wedge dz)$; $*(dy \wedge dt)$; $*(dz \wedge dt)$.

We have, respectively,

- $*1 = dx \wedge dy \wedge dz \wedge dt$;
- $*dx = dy \wedge dz \wedge dt$;
- $*dy = -dx \wedge dz \wedge dt$;
- $*dz = dx \wedge dy \wedge dt$;
- $*dt = dx \wedge dy \wedge dz$;
- $*(dx \wedge dy) = dz \wedge dt$;
- $*(dx \wedge dz) = -dy \wedge dt$;
- $*(dx \wedge dt) = -dy \wedge dz$;

- $*(dy \wedge dz) = dx \wedge dt$;
- $*(dy \wedge dt) = dx \wedge dz$;
- $*(dz \wedge dt) = -dx \wedge dy$.

Definition. We define the electromagnetic *Faraday* 2*-form* $F \in \mathrm{T}^*\mathrm{M} \otimes_A \mathrm{T}^*\mathrm{M}$ as follows:

$$F = -E_1\, dx \wedge dt - E_2\, dy \wedge dt - E_3\, dz \wedge dt - B_1\, dy \wedge dz + B_2\, dx \wedge dz - B_3\, dx \wedge dy \stackrel{\text{def}}{=} \Phi(\boldsymbol{B}, \boldsymbol{E})\,.$$

The independent components E_i and B_i ($i = 1, 2, 3$) may be collected into 3-vectors $\boldsymbol{E} = (E_1, E_2, E_3)$ and $\boldsymbol{B} = (B_1, B_2, B_3)$. For $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, $\mathbf{u} = (u, v, w) \in \mathbb{R}^3$, standard norm and inner product are denoted by $\|\mathbf{x}\|$ and $\mathbf{x} \cdot \mathbf{y}$, respectively, with $\mathbf{x} \cdot \mathbf{u} = xu + yv + zw$ and $\|\mathbf{x}\| = (x^2 + y^2 + z^2)^{1/2}$.

(20) **b.** Compute the following tensor fields. Express your results in terms of the 3-vectors $\boldsymbol{B}$ and $\boldsymbol{E}$ and derived quantities, such as $\|\boldsymbol{B}\|$, $\|\boldsymbol{E}\|$ and $\boldsymbol{B} \cdot \boldsymbol{E}$:

b1. $*F$.
[HINT: WRITE $*F \stackrel{\text{DEF}}{=} \Psi(\boldsymbol{B}, \boldsymbol{E})$. PROVE $\Psi(\boldsymbol{B}, \boldsymbol{E}) = \Phi(-\boldsymbol{E}, \boldsymbol{B})$.]

Linearity of $*$ in combination with the results of problem a3 reveals that $\Psi(\boldsymbol{B}, \boldsymbol{E}) = \Phi(-\boldsymbol{E}, \boldsymbol{B})$.

b2. $F \wedge F$.

Using antisymmetry of the $\wedge$ product to rearrange factors in an exterior product, in particular $dx \wedge dx = dy \wedge dy = dz \wedge dz = dt \wedge dt = 0$, the only nontrivial product terms are proportional to $\boldsymbol{\epsilon} = dx \wedge dy \wedge dz \wedge dt$. A straightforward calculation, based on the results of a3, yields

$$F \wedge F = \Phi(\boldsymbol{B}, \boldsymbol{E}) \wedge \Phi(\boldsymbol{B}, \boldsymbol{E}) = 2(B_1E_1 + B_2E_2 + B_3E_3)\, dx \wedge dy \wedge dz \wedge dt = 2\boldsymbol{B} \cdot \boldsymbol{E}\, \boldsymbol{\epsilon}\,.$$

b3. $F \wedge *F$.

Using the definition of F and the calculation of $*F$ from b1 we find, by a calculation similar to the one in b2,

$$F \wedge *F = \Phi(\boldsymbol{B}, \boldsymbol{E}) \wedge \Psi(\boldsymbol{B}, \boldsymbol{E}) = \Phi(\boldsymbol{B}, \boldsymbol{E}) \wedge \Phi(-\boldsymbol{E}, \boldsymbol{B}) = (B_1^2 + B_2^2 + B_3^2 - E_1^2 - E_2^2 - E_3^2)\, dx \wedge dy \wedge dz \wedge dt = (\|\boldsymbol{B}\|^2 - \|\boldsymbol{E}\|^2)\, \boldsymbol{\epsilon}\,.$$

♣

(40) **3. HESSIAN OF A SCALAR FIELD WITH AND WITHOUT RIEMANNIAN METRIC.**

Definition. On a differentiable manifold M, torsion is defined in terms of a third order mixed tensor field $\mathbf{t} \in \mathrm{T}^*\mathrm{M} \otimes \mathrm{T}^*\mathrm{M} \otimes \mathrm{TM}$, viz. if $\mathbf{v}, \mathbf{w} \in \mathrm{TM}$ and $\boldsymbol{\omega} \in \mathrm{T}^*\mathrm{M}$ are vector and covector fields, respectively, then $\mathbf{t}(\mathbf{v}, \mathbf{w}, \boldsymbol{\omega}) = \langle \boldsymbol{\omega}, \mathbf{T}(\mathbf{v}, \mathbf{w}) \rangle$, in which

$$\mathbf{T}(\mathbf{v}, \mathbf{w}) = \nabla_{\mathbf{v}} \mathbf{w} - \nabla_{\mathbf{w}} \mathbf{v} - [\mathbf{v}, \mathbf{w}] \,.$$

Definition. The Hessian $\mathbf{H}(f)$ of a scalar field $f \in C^2(\mathrm{M})$ on M is defined as a second order cotensor field $\mathbf{H}(f) \in \mathrm{T}^*\mathrm{M} \otimes \mathrm{T}^*\mathrm{M}$. Given two vector fields $\mathbf{v}, \mathbf{w} \in \mathrm{TM}$ we have, by definition,

$$\mathbf{H}(f)(\mathbf{v}, \mathbf{w}) = \nabla_{\mathbf{v}} \nabla_{\mathbf{w}} f - \nabla_{\nabla_{\mathbf{v}} \mathbf{w}} f \,,$$

in which ∇ denotes an affine connection.

(10) **a.** Show that, in general, $\mathbf{H}(f)$ is not symmetric, and explain the role of $\mathbf{T}$ in this matter.

Using $\nabla_{\mathbf{x}} f = \mathbf{x} f$ for any vector field $\mathbf{x} \in \mathrm{TM}$ and scalar function $f \in C^2(\mathrm{M})$, we have

$$\mathbf{H}(f)(\mathbf{v}, \mathbf{w}) - \mathbf{H}(f)(\mathbf{w}, \mathbf{v}) = \nabla_{\mathbf{v}} \nabla_{\mathbf{w}} f - \nabla_{\nabla_{\mathbf{v}} \mathbf{w}} f - \nabla_{\mathbf{w}} \nabla_{\mathbf{v}} f + \nabla_{\nabla_{\mathbf{w}} \mathbf{v}} f = \mathbf{v}\mathbf{w} f - \nabla_{\mathbf{v}} \mathbf{w} f - \mathbf{w}\mathbf{v} f + \nabla_{\mathbf{w}} \mathbf{v} f = -\mathbf{T}(\mathbf{v}, \mathbf{w}) f \,.$$

Assumption. We henceforth assume M to be Riemannian, with torsion free Levi-Civita connection ∇.

(10) **b.** Show that, in this case, $\mathbf{H}(f)$ *is* symmetric.

For the Levi-Civita connection we have, by definition, $\mathbf{T} = \mathbf{0}$, so that $\mathbf{H}(f)(\mathbf{v}, \mathbf{w}) = \mathbf{H}(f)(\mathbf{w}, \mathbf{v})$, recall a.

Definition. Relative to a coordinate basis $\{\partial_i\}$, $i = 1, \ldots, n$, we have $\nabla_{\partial_j} \partial_i = \Gamma^k_{ij} \partial_k$.

(10) **c.** Compute the holor $H_{ij}(f) \stackrel{\text{def}}{=} \mathbf{H}(f)(\partial_i, \partial_j)$ of $\mathbf{H}(f)$ in terms of the (holor of the) Riemannian metric $g_{ij} = (\partial_i | \partial_j)$ and/or its dual, and the Christoffel symbols Γ^k_{ij}.

We have, again making repetitive use of the identity $\nabla_{\mathbf{x}} f = \mathbf{x} f$,

$$H_{ij}(f) = \mathbf{H}(f)(\partial_i, \partial_j) = \nabla_{\partial_i} \nabla_{\partial_j} f - \nabla_{\nabla_{\partial_i} \partial_j} f = \partial_i \partial_j f - \Gamma^k_{ji} \partial_k f \,.$$

Definition. The Laplacian of a scalar field f is the trace of its Hessian: $\Delta f \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{H}(f) \stackrel{\text{def}}{=} g^{ij} H_{ij}(f)$.

(10) **d.** Show that $\Delta f = \dfrac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f \right)$.

[HINT: RECALL $\partial_i g = g g^{k\ell} \partial_i g_{k\ell}$. YOU MAY NEED THE TRIVIAL OBSERVATION $\partial_i (g^{jk} g_{k\ell}) = 0$.]

Using the product rule we may expand

$$\Delta f = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f \right) = g^{ij} \partial_i \partial_j f + \frac{1}{2g} \partial_i g \, g^{ij} \partial_j f + \partial_i g^{ij} \partial_j f \,.$$

Following the hint we may rewrite the second and third term as follows. Substituting $\partial_i g = g g^{k\ell} \partial_i g_{k\ell}$ and $\partial_i g^{ij} = -g^{ik} g^{\ell j} \partial_i g_{k\ell}$ into these terms, we observe, using some elementary dummy relabelings (using symmetry of the metric holor), that

$$\frac{1}{2g} \partial_i g g^{ij} \partial_j f + \partial_i g^{ij} \partial_j f = g^{ij} \left\{ \frac{1}{2} g^{k\ell} \left(\partial_\ell g_{ij} - \partial_i g_{\ell j} - \partial_j g_{i\ell} \right) \right\} \partial_k f = -g^{ij} \Gamma^k_{ji} \partial_k f \,.$$

THE END