# EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY 

Course code: 2WAH0. Date: Thursday June 27, 2017. Time: 18h00-21h00. Place: AUD 15.

## Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- Einstein summation convention applies throughout for all repeated indices.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No equipment may be used.



## 1. Strain in Euclidean Space.

Definition. The Lie derivative of a smooth scalar field $f \in C^{\infty}\left(\mathbb{R}^{n}\right)$ with respect to a smooth vector field $\mathbf{v} \in \mathbb{T R}^{n}$ is given by its directional derivative

$$
\mathscr{L}_{\mathbf{v}} f=\mathbf{v} f,
$$

with $\mathbf{v} f=v^{k} \partial_{k} f$.
Definition. The Lie derivative of a smooth covector field $\boldsymbol{\omega} \in \mathrm{T}^{*} \mathbb{R}^{n}$ with respect to a smooth vector field $\mathbf{v} \in \mathbb{R}^{n}$ is defined as

$$
\mathscr{L}_{\mathbf{v}} \boldsymbol{\omega}=d\langle\boldsymbol{\omega}, \mathbf{v}\rangle,
$$

in which $\langle,\rangle \in \mathrm{T}^{*} \mathbb{R}^{n} \otimes \mathbb{T}^{n}$ is the Kronecker tensor field and $d$ denotes the differential with respect to the (implicit) basepoint coordinate chart $x \in \mathbb{R}^{n}$, i.e. $d f=\partial_{k} f d x^{k}$ for any $f \in C^{\infty}\left(\mathbb{R}^{n}\right)$.

Definition. The Lie derivative is extended to general cotensors $\mathbf{T} \in \mathrm{T}^{*} \mathbb{R}^{n} \otimes \ldots \otimes \mathrm{~T}^{*} \mathbb{R}^{n}$ via linearity and product rule in the usual way. For example, if $\mathbf{g}=g_{i j} d x^{i} \otimes d x^{j} \in \mathrm{~T}^{*} \mathbb{R}^{n} \otimes \mathbf{T}^{*} \mathbb{R}^{n}$ with smooth coefficient functions $g_{i j} \in C^{\infty}\left(\mathbb{R}^{n}\right)$, then $\mathscr{L}_{\mathbf{v}} \mathbf{g}=\left(\mathscr{L}_{\mathbf{v}} g_{i j}\right) d x^{i} \otimes d x^{j}+g_{i j}\left(\mathscr{L}_{\mathbf{v}} d x^{i}\right) \otimes d x^{j}+g_{i j} d x^{i} \otimes\left(\mathscr{L}_{\mathbf{v}} d x^{j}\right)$.

Terminology. If $\mathbf{g}$ is the metric tensor field, with holor $g_{i j}=\left(\partial_{i} \mid \partial_{j}\right)$, then $\mathscr{L}_{\mathbf{v}} \mathbf{g}$ is known as the strain tensor field with respect to the displacement field $\mathbf{v}$ induced by a deformation of the medium.
a. Compute the holor of the strain tensor field $\mathscr{L}_{\mathrm{v}} \mathrm{g}$ in a general coordinate chart.
b. Show that in Cartesian coordinates the strain tensor field simplifies to $\mathscr{L}_{\mathbf{v}} \mathbf{g}=\left(\partial_{i} v_{j}+\partial_{j} v_{i}\right) d x^{i} \otimes d x^{j}$.

Definition. We consider 4-dimensional Minkowski spacetime M, furnished with a Lorentzian metric $(\mathbf{x} \mid \mathbf{y})=G(\mathbf{x}, \mathbf{y})=g_{i j} x^{i} y^{j}$. The components of the corresponding Gram matrix $\mathbf{G}$ are $g_{i j}=\left(\partial_{i} \mid \partial_{j}\right)$ $(i, j=1,2,3,4)$. We abbreviate $g=\operatorname{det} \mathbf{G}$.

Definition. We employ canonical coordinates, in terms of which $g_{11}=g_{22}=g_{33}=1, g_{44}=-1, g_{i j}=0$ otherwise. We abbreviate $\partial_{1}=\partial_{x}, \partial_{2}=\partial_{y}, \partial_{3}=\partial_{z}, \partial_{4}=\partial_{t}, d x^{1}=d x, d x^{2}=d y, d x^{3}=d z, d x^{4}=d t$.

Definition. The Levi-Civita tensor in $n$ dimensions is given by

$$
\boldsymbol{\epsilon}=\sqrt{|g|} d x^{1} \wedge \ldots \wedge d x^{n}=\epsilon_{\left|i_{1} \ldots i_{n}\right|} d x^{i_{1}} \wedge \ldots \wedge d x^{i_{n}} .
$$

Definition. The linear Hodge star operator $*: \mathrm{T}^{*} \mathrm{M} \underbrace{\otimes_{A} \ldots \otimes_{A}}_{k} \mathrm{~T}^{*} \mathrm{M} \rightarrow \mathrm{T}^{*} \mathrm{M} \underbrace{\otimes_{A} \ldots \otimes_{A}}_{n-k} \mathrm{~T}^{*} \mathrm{M}$ satisfies

$$
*\left(d x^{i_{1}} \wedge \ldots \wedge d x^{i_{k}}\right)=\frac{1}{(n-k)!} g^{i_{1} j_{1}} \ldots g^{i_{k} j_{k}} \epsilon_{j_{1} \ldots j_{n}} d x^{j_{k+1}} \wedge \ldots \wedge d x^{j_{n}} .
$$

a. Compute the following forms in terms of wedge products of $d x, d y, d z$, and $d t$ :
a1. $* 1$.
a2. $* d x ; * d y ; * d z ; * d t$.
a3. $*(d x \wedge d y) ; *(d x \wedge d z) ; *(d x \wedge d t) ; *(d y \wedge d z) ; *(d y \wedge d t) ; *(d z \wedge d t)$.
Definition. We define the electromagnetic Faraday 2-form $F \in \mathrm{~T}^{*} \mathrm{M} \otimes_{A} \mathrm{~T}^{*} \mathrm{M}$ as follows:
$F=-E_{1} d x \wedge d t-E_{2} d y \wedge d t-E_{3} d z \wedge d t-B_{1} d y \wedge d z+B_{2} d x \wedge d z-B_{3} d x \wedge d y \stackrel{\text { def }}{=} \Phi(\boldsymbol{B}, \boldsymbol{E})$.
The independent components $E_{i}$ and $B_{i}(i=1,2,3)$ may be collected into 3-vectors $\boldsymbol{E}=\left(E_{1}, E_{2}, E_{3}\right)$ and $\boldsymbol{B}=\left(B_{1}, B_{2}, B_{3}\right)$. For $\mathbf{x}=(x, y, z) \in \mathbb{R}^{3}, \mathbf{u}=(u, v, w) \in \mathbb{R}^{3}$, standard norm and inner product are denoted by $\|\mathbf{x}\|$ and $\mathbf{x} \cdot \mathbf{y}$, respectively, with $\mathbf{x} \cdot \mathbf{u}=x u+y v+z w$ and $\|\mathbf{x}\|=\left(x^{2}+y^{2}+z^{2}\right)^{1 / 2}$.
(20) b. Compute the following tensor fields. Express your results in terms of the 3 -vectors $\boldsymbol{B}$ and $\boldsymbol{E}$ and derived quantities, such as $\|\boldsymbol{B}\|,\|\boldsymbol{E}\|$ and $\boldsymbol{B} \cdot \boldsymbol{E}$ :
b1. $* F$.
[Hint: Write $* F \stackrel{\text { DEF }}{=} \Psi(\boldsymbol{B}, \boldsymbol{E})$. $\operatorname{Prove} \Psi(\boldsymbol{B}, \boldsymbol{E})=\Phi(-\boldsymbol{E}, \boldsymbol{B})$.]
b2. $F \wedge F$.
b3. $F \wedge * F$.

Definition. On a differentiable manifold M , torsion is defined in terms of a third order mixed tensor field $\mathbf{t} \in \mathrm{T}^{*} \mathrm{M} \otimes \mathrm{T}^{*} \mathrm{M} \otimes \mathrm{TM}$, viz. if $\mathbf{v}, \mathbf{w} \in \mathrm{TM}$ and $\boldsymbol{\omega} \in \mathrm{T}^{*} \mathrm{M}$ are vector and covector fields, respectively, then $\mathbf{t}(\mathbf{v}, \mathbf{w}, \boldsymbol{\omega})=\langle\boldsymbol{\omega}, \mathbf{T}(\mathbf{v}, \mathbf{w})\rangle$, in which

$$
\mathbf{T}(\mathbf{v}, \mathbf{w})=\nabla_{\mathbf{v}} \mathbf{w}-\nabla_{\mathbf{w}} \mathbf{v}-[\mathbf{v}, \mathbf{w}] .
$$

Definition. The Hessian $\mathbf{H}(f)$ of a scalar field $f \in C^{2}(\mathbf{M})$ on $\mathbf{M}$ is defined as a second order cotensor field $\mathbf{H}(f) \in \mathrm{T}^{*} \mathbf{M} \otimes \mathrm{~T}^{*} \mathbf{M}$. Given two vector fields $\mathbf{v}, \mathbf{w} \in \mathrm{TM}$ we have, by definition,

$$
\mathbf{H}(f)(\mathbf{v}, \mathbf{w})=\nabla_{\mathbf{v}} \nabla_{\mathbf{w}} f-\nabla_{\nabla_{\mathbf{v}}} f,
$$

in which $\nabla$ denotes an affine connection.
(10) a. Show that, in general, $\mathbf{H}(f)$ is not symmetric, and explain the role of $\mathbf{T}$ in this matter.

Assumption. We henceforth assume M to be Riemannian, with torsion free Levi-Civita connection $\nabla$.
(10) b. Show that, in this case, $\mathbf{H}(f)$ is symmetric.

Definition. Relative to a coordinate basis $\left\{\partial_{i}\right\}, i=1, \ldots, n$, we have $\nabla_{\partial_{j}} \partial_{i}=\Gamma_{i j}^{k} \partial_{k}$.
(10) c. Compute the holor $H_{i j}(f) \stackrel{\text { def }}{=} \mathbf{H}(f)\left(\partial_{i}, \partial_{j}\right)$ of $\mathbf{H}(f)$ in terms of the (holor of the) Riemannian metric $g_{i j}=\left(\partial_{i} \mid \partial_{j}\right)$ and/or its dual, and the Christoffel symbols $\Gamma_{i j}^{k}$.

Definition. The Laplacian of a scalar field $f$ is the trace of its Hessian: $\Delta f \stackrel{\text { def }}{=} \operatorname{tr} \mathbf{H}(f) \stackrel{\text { def }}{=} g^{i j} H_{i j}(f)$.
(10) d. Show that $\Delta f=\frac{1}{\sqrt{g}} \partial_{i}\left(\sqrt{g} g^{i j} \partial_{j} f\right)$.
[Hint: Recall $\partial_{i} g=g g^{k \ell} \partial_{i} g_{k \ell}$. You may need the trivial observation $\partial_{i}\left(g^{j k} g_{k \ell}\right)=0$.]

## The End

