EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY

Course code: 2WAH0. Date: Thursday June 27, 2017. Time: 18h00–21h00. Place: AUD 15.

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in the margin.
- Einstein summation convention applies throughout for all repeated indices.
- You may consult an immaculate hardcopy of the online draft notes "Tensor Calculus and Differential Geometry (2WAH0)" by Luc Florack. No equipment may be used.



(20) 1. STRAIN IN EUCLIDEAN SPACE.

Definition. The Lie derivative of a smooth scalar field $f \in C^{\infty}(\mathbb{R}^n)$ with respect to a smooth vector field $\mathbf{v} \in T\mathbb{R}^n$ is given by its directional derivative

$$\mathscr{L}_{\mathbf{v}}f=\mathbf{v}f\,,$$

with $\mathbf{v}f = v^k \partial_k f$.

Definition. The Lie derivative of a smooth covector field $\omega \in T^*\mathbb{R}^n$ with respect to a smooth vector field $\mathbf{v} \in T\mathbb{R}^n$ is defined as

$$\mathscr{L}_{\mathbf{v}}\boldsymbol{\omega} = d\langle \boldsymbol{\omega}, \mathbf{v} \rangle,$$

in which $\langle , \rangle \in T^* \mathbb{R}^n \otimes T \mathbb{R}^n$ is the Kronecker tensor field and d denotes the differential with respect to the (implicit) basepoint coordinate chart $x \in \mathbb{R}^n$, i.e. $df = \partial_k f dx^k$ for any $f \in C^{\infty}(\mathbb{R}^n)$.

Definition. The Lie derivative is extended to general cotensors $\mathbf{T} \in \mathbf{T}^* \mathbb{R}^n \otimes \ldots \otimes \mathbf{T}^* \mathbb{R}^n$ via linearity and product rule in the usual way. For example, if $\mathbf{g} = g_{ij} dx^i \otimes dx^j \in \mathbf{T}^* \mathbb{R}^n \otimes \mathbf{T}^* \mathbb{R}^n$ with smooth coefficient functions $g_{ij} \in C^{\infty}(\mathbb{R}^n)$, then $\mathscr{L}_{\mathbf{v}}\mathbf{g} = (\mathscr{L}_{\mathbf{v}}g_{ij}) dx^i \otimes dx^j + g_{ij} (\mathscr{L}_{\mathbf{v}}dx^i) \otimes dx^j + g_{ij} dx^i \otimes (\mathscr{L}_{\mathbf{v}}dx^j)$.

Terminology. If g is the metric tensor field, with holor $g_{ij} = (\partial_i | \partial_j)$, then $\mathscr{L}_{\mathbf{v}} \mathbf{g}$ is known as the *strain tensor field* with respect to the displacement field v induced by a deformation of the medium.

- (10) **a.** Compute the holor of the strain tensor field $\mathscr{L}_{\mathbf{v}}\mathbf{g}$ in a general coordinate chart.
- (10) **b.** Show that in Cartesian coordinates the strain tensor field simplifies to $\mathscr{L}_{\mathbf{v}}\mathbf{g} = (\partial_i v_j + \partial_j v_i) dx^i \otimes dx^j$.

(40) 2. HODGE STAR AND FARADAY 2-FORM IN MINKOWSKI SPACETIME.

Definition. We consider 4-dimensional Minkowski spacetime M, furnished with a Lorentzian metric $(\mathbf{x}|\mathbf{y}) = G(\mathbf{x}, \mathbf{y}) = g_{ij}x^iy^j$. The components of the corresponding Gram matrix **G** are $g_{ij} = (\partial_i|\partial_j)$ (i, j = 1, 2, 3, 4). We abbreviate $g = \det \mathbf{G}$.

Definition. We employ canonical coordinates, in terms of which $g_{11} = g_{22} = g_{33} = 1$, $g_{44} = -1$, $g_{ij} = 0$ otherwise. We abbreviate $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\partial_3 = \partial_z$, $\partial_4 = \partial_t$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$, $dx^4 = dt$.

Definition. The Levi-Civita tensor in n dimensions is given by

$$\boldsymbol{\epsilon} = \sqrt{|g|} \, dx^1 \wedge \ldots \wedge dx^n = \epsilon_{|i_1 \dots i_n|} dx^{i_1} \wedge \ldots \wedge dx^{i_n} \, .$$

Definition. The linear Hodge star operator $*: T^*M \underbrace{\otimes_{A \dots \otimes_{A}}}_{k} T^*M \to T^*M \underbrace{\otimes_{A \dots \otimes_{A}}}_{n-k} T^*M$ satisfies

$$*\left(dx^{i_1}\wedge\ldots\wedge dx^{i_k}\right) = \frac{1}{(n-k)!}g^{i_1j_1}\ldots g^{i_kj_k}\epsilon_{j_1\ldots j_n}dx^{j_{k+1}}\wedge\ldots\wedge dx^{j_n}$$

- (20) **a.** Compute the following forms in terms of wedge products of dx, dy, dz, and dt:
 - **a1.** *1.
 - **a2.** *dx; *dy; *dz; *dt.

a3.
$$*(dx \wedge dy)$$
; $*(dx \wedge dz)$; $*(dx \wedge dt)$; $*(dy \wedge dz)$; $*(dy \wedge dt)$; $*(dz \wedge dt)$.

Definition. We define the electromagnetic *Faraday* 2-*form* $F \in T^*M \otimes_A T^*M$ as follows:

$$F = -E_1 \, dx \wedge dt - E_2 \, dy \wedge dt - E_3 \, dz \wedge dt - B_1 \, dy \wedge dz + B_2 \, dx \wedge dz - B_3 \, dx \wedge dy \stackrel{\text{def}}{=} \Phi(\boldsymbol{B}, \boldsymbol{E}) \, dz + B_2 \, dz \wedge dz - B_3 \, dz \wedge dy \stackrel{\text{def}}{=} \Phi(\boldsymbol{B}, \boldsymbol{E}) \, dz + B_2 \, dz \wedge dz - B_3 \, dz \wedge dy \stackrel{\text{def}}{=} \Phi(\boldsymbol{B}, \boldsymbol{E}) \, dz + B_2 \, dz \wedge dz - B_3 \, dz \wedge dy \stackrel{\text{def}}{=} \Phi(\boldsymbol{B}, \boldsymbol{E}) \, dz$$

The independent components E_i and B_i (i = 1, 2, 3) may be collected into 3-vectors $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$. For $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, $\mathbf{u} = (u, v, w) \in \mathbb{R}^3$, standard norm and inner product are denoted by $\|\mathbf{x}\|$ and $\mathbf{x} \cdot \mathbf{y}$, respectively, with $\mathbf{x} \cdot \mathbf{u} = xu + yv + zw$ and $\|\mathbf{x}\| = (x^2 + y^2 + z^2)^{1/2}$.

(20) **b.** Compute the following tensor fields. Express your results in terms of the 3-vectors B and E and derived quantities, such as ||B||, ||E|| and $B \cdot E$:

b1. **F*. [Hint: Write * $F \stackrel{\text{def}}{=} \Psi(\boldsymbol{B}, \boldsymbol{E})$. Prove $\Psi(\boldsymbol{B}, \boldsymbol{E}) = \Phi(-\boldsymbol{E}, \boldsymbol{B})$.]

b2. $F \wedge F$.

b3. $F \wedge *F$.

(40) 3. HESSIAN OF A SCALAR FIELD WITH AND WITHOUT RIEMANNIAN METRIC.

Definition. On a differentiable manifold M, torsion is defined in terms of a third order mixed tensor field $\mathbf{t} \in T^*M \otimes T^*M \otimes TM$, viz. if $\mathbf{v}, \mathbf{w} \in TM$ and $\boldsymbol{\omega} \in T^*M$ are vector and covector fields, respectively, then $\mathbf{t}(\mathbf{v}, \mathbf{w}, \boldsymbol{\omega}) = \langle \boldsymbol{\omega}, \mathbf{T}(\mathbf{v}, \mathbf{w}) \rangle$, in which

$$\mathbf{T}(\mathbf{v}, \mathbf{w}) = \nabla_{\mathbf{v}} \mathbf{w} - \nabla_{\mathbf{w}} \mathbf{v} - [\mathbf{v}, \mathbf{w}] \ .$$

Definition. The Hessian $\mathbf{H}(f)$ of a scalar field $f \in C^2(\mathbf{M})$ on \mathbf{M} is defined as a second order cotensor field $\mathbf{H}(f) \in T^*\mathbf{M} \otimes T^*\mathbf{M}$. Given two vector fields $\mathbf{v}, \mathbf{w} \in T\mathbf{M}$ we have, by definition,

$$\mathbf{H}(f)(\mathbf{v}, \mathbf{w}) = \nabla_{\mathbf{v}} \nabla_{\mathbf{w}} f - \nabla_{\nabla_{\mathbf{v}} \mathbf{w}} f,$$

in which ∇ denotes an affine connection.

(10) **a.** Show that, in general, $\mathbf{H}(f)$ is not symmetric, and explain the role of **T** in this matter.

Assumption. We henceforth assume M to be Riemannian, with torsion free Levi-Civita connection ∇ .

(10) **b.** Show that, in this case, $\mathbf{H}(f)$ is symmetric.

Definition. Relative to a coordinate basis $\{\partial_i\}, i=1,\ldots,n$, we have $\nabla_{\partial_i}\partial_i = \Gamma_{ij}^k\partial_k$.

(10) **c.** Compute the holor $H_{ij}(f) \stackrel{\text{def}}{=} \mathbf{H}(f)(\partial_i, \partial_j)$ of $\mathbf{H}(f)$ in terms of the (holor of the) Riemannian metric $g_{ij} = (\partial_i | \partial_j)$ and/or its dual, and the Christoffel symbols Γ_{ij}^k .

Definition. The Laplacian of a scalar field f is the trace of its Hessian: $\Delta f \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{H}(f) \stackrel{\text{def}}{=} g^{ij} H_{ij}(f)$.

(10) **d.** Show that $\Delta f = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f \right).$ [HINT: RECALL $\partial_i g = g g^{k\ell} \partial_i g_{k\ell}$. You may need the trivial observation $\partial_i (g^{jk} g_{k\ell}) = 0.$]

THE END