# EXAMINATION TENSOR CALCULUS AND DIFFERENTIAL GEOMETRY 

Course code: 2WAHO. Date: Tuesday June 30, 2020. Time: 18h00-21h00. Canvas online assignment.

READ THIS FIRST!

- Write your name and student identification number on each paper, and include it in the file name(s) of any scan you upload via Assignments in Canvas.
- The exam consists of 3 problems and a mandatory legal statement. Credits are indicated in the margin.
- The Einstein summation convention is in effect throughout this exam.
- Follow the legal statement instruction at the end of the exam before you scan and upload your results. This requires you to include a written statement exactly as indicated in that clause.
- You may consult all material provided specifically for "Tensor Calculus \& Differential Geometry (2WAH0)" by Luc Florack in the Files folder of Canvas.
- Keep your camera on and do not unmute yourself.
- You can chat with the host/invigilator (privately) for urgent requests. Do not step away from the camera without request and explicit approval. A short sanitary break will be allowed during the exam.


## 1. Tensor Calculus \& Differential Geometry Miscellany

(7 $7 \frac{1}{2}$ ) a. Prove that the volume form $\sqrt{g} d x^{1} \wedge \ldots \wedge d x^{n}$ on an $n$-dimensional Riemannian manifold M is 'almost' invariant under coordinate transformations. Explain 'almost'.

We have

$$
d \bar{x}^{1} \wedge \ldots \wedge d \bar{x}^{n}=\frac{\partial \bar{x}^{1}}{\partial x^{i_{1}}} \ldots \frac{\partial \bar{x}^{n}}{\partial x^{i_{n}}} d x^{i_{1}} \wedge \ldots \wedge d x^{i_{n}}=\left[i_{1} \ldots i_{n}\right] \frac{\partial \bar{x}^{1}}{\partial x^{i_{1}}} \ldots \frac{\partial \bar{x}^{n}}{\partial x^{i_{n}}} d x^{1} \wedge \ldots \wedge d x^{n}=\operatorname{det} \frac{\partial \bar{x}}{\partial x} d x^{1} \wedge \ldots \wedge d x^{n} .
$$

Also,

$$
\bar{g}=\operatorname{det} \bar{g}_{\bullet \bullet}=\operatorname{det}\left(\frac{\partial x^{k}}{\partial \bar{x}^{\bullet}} \frac{\partial x^{\ell}}{\partial \bar{x}^{\bullet}} g_{k \ell}\right)=\left(\operatorname{det} \frac{\partial x}{\partial \bar{x}}\right)^{2} g=\left(\operatorname{det} \frac{\partial \bar{x}}{\partial x}\right)^{-2} g .
$$

Therefore $\sqrt{\bar{g}} d \bar{x}^{1} \wedge \ldots \wedge d \bar{x}^{n}=\operatorname{sgn}\left(\operatorname{det} \frac{\partial \bar{x}}{\partial x}\right) d x^{1} \wedge \ldots \wedge d x^{n}$. This shows that the volume form is 'almost' invariant in the sense that it will at most reverse sign, viz. under an orientation reversing basis transformation.

The Lie derivative of a 1 -form field $\boldsymbol{\omega}$ with respect to a vector field $\mathbf{v}$ is the 1 -form field $\mathscr{L}_{\mathbf{v}} \boldsymbol{\omega} \doteq d\langle\boldsymbol{\omega}, \mathbf{v}\rangle$, in which $\langle\bullet, \bullet\rangle$ denotes the Kronecker tensor. The Lie derivative of a scalar field $f$ with respect to a vector field $\mathbf{v}$ is defined as the directional derivative $\mathscr{L}_{\mathbf{v}} \doteq \mathbf{v} f$. Lie derivatives act linearly on their operands and satisfy the product rule when applied to tensor products.

Definition. The strain tensor on a metric space M with metric $\mathbf{g}=g_{i j} d x^{i} \otimes d x^{j}$ with respect to a displacement vector field $\mathbf{v}$ is defined as $\mathscr{L}_{\mathbf{v}} \mathbf{g}$.
b. Compute the components of $\mathscr{L}_{\mathbf{v}} \mathbf{g}$ relative to a coordinate basis, with $\mathbf{v} \doteq v^{i} \partial_{i}$.

We have

$$
\mathscr{L}_{\mathbf{v}} \mathbf{g}=\mathscr{L}_{\mathbf{v}}\left(g_{i j} d x^{i} \otimes d x^{j}\right)=\mathscr{L}_{\mathbf{v}} g_{i j} d x^{i} \otimes d x^{j}+g_{i j} \mathscr{L}_{\mathbf{v}} d x^{i} \otimes d x^{j}+g_{i j} d x^{i} \otimes \mathscr{L}_{\mathbf{v}} d x^{j} .
$$

Using $\mathscr{L}_{\mathbf{v}} g_{i j}=v^{k} \partial_{k} g_{i j}$ and $\mathscr{L}_{\mathbf{v}} d x^{i}=d\left\langle d x^{i}, \mathbf{v}\right\rangle=d v^{i}=\partial_{k} v^{i} d x^{k}$, we find, after some trivial dummy index manipulations to factorise a common basis $d x^{i} \otimes d x^{j}$,

$$
\mathscr{L}_{\mathbf{v}} \mathbf{g}=\left(v^{k} \partial_{k} g_{i j}+g_{k j} \partial_{i} v^{k}+g_{i k} \partial_{j} v^{k}\right) d x^{i} \otimes d x^{j}
$$

Let $\Gamma_{j k}^{\ell}$ be the Christoffel symbols associated with $\mathbf{g}$, and $g \doteq \operatorname{det} \mathbf{g}$ the determinant of the Gram matrix.
c. Prove: $\Gamma_{\ell k}^{\ell}=\partial_{k} \ln \sqrt{g}$.

Contracting the indices $\ell$ and $m$ in the defining formula

$$
\Gamma_{k \ell}^{m}=\frac{1}{2} g^{m n}\left(\partial_{k} g_{n \ell}+\partial_{\ell} g_{k n}-\partial_{n} g_{k \ell}\right)
$$

yields

$$
\Gamma_{k \ell}^{\ell}=\frac{1}{2} g^{\ell n}\left(\partial_{k} g_{n \ell}+\partial_{\ell} g_{k n}-\partial_{n} g_{k \ell}\right) \stackrel{*}{=} \frac{1}{2} g^{\ell n} \partial_{k} g_{n \ell}
$$

in which $*$ follows from the observation that the last two terms inside parenthesis are antisymmetric w.r.t. $\ell$ and $n$, whereas $g^{\ell n}$ is symmetric. On the other hand we have the definition of the (symmetric) cofactor matrix

$$
\widetilde{g}^{i j} \doteq \frac{\partial g}{\partial g_{i j}}
$$

besides the identity $g^{i j}=\frac{1}{g} \widetilde{g}^{i j}$, from which it follows, with the help of the chain rule and symmetry of the metric tensor, that

$$
\partial_{k} \ln \sqrt{g}=\frac{1}{\sqrt{g}} \partial_{k} \sqrt{g}=\frac{1}{2 g} \partial_{k} g=\frac{1}{2 g} \frac{\partial g}{\partial g_{\ell n}} \partial_{k} g_{\ell n}=\frac{1}{2} g^{\ell n} \partial_{k} g_{n \ell}=\Gamma_{k \ell}^{\ell}
$$

The last equality has been proven above.
Let $A^{i j}$ be the holor of an antisymmetric contravariant tensor field of rank 2.
$\left(7 \frac{1}{2}\right) \quad$ d. Prove: $D_{k} A^{i k}=\frac{1}{\sqrt{g}} \partial_{k}\left(\sqrt{g} A^{i k}\right)$.

We have

$$
D_{k} A^{i j}=\partial_{k} A^{i j}+\Gamma_{\ell k}^{i} A^{\ell j}+\Gamma_{\ell k}^{j} A^{i \ell}
$$

Contracting the indices $j$ and $k$ yields

$$
D_{k} A^{i k}=\partial_{k} A^{i k}+\Gamma_{\ell k}^{i} A^{\ell k}+\Gamma_{\ell k}^{k} A^{i \ell} \stackrel{\star}{=} \partial_{k} A^{i k}+\Gamma_{\ell k}^{k} A^{i \ell}
$$

in which $\star$ follows from the fact that $A^{i k}$ is antisymmetric, whereas $\Gamma_{\ell k}^{i}$ is symmetric w.r.t. $k$ and $\ell$. Inserting $\Gamma_{\ell k}^{k}=\partial_{\ell} \ln \sqrt{g}$ as obtained in c , besides elementary dummy index manipulations, allows us to rewrite this as

$$
D_{k} A^{i k}=\partial_{k} A^{i k}+\partial_{\ell} \ln \sqrt{g} A^{i \ell}=\frac{1}{\sqrt{g}} \partial_{k}\left(\sqrt{g} A^{i k}\right)
$$

## (35) 2. TORSION

Based on a given affine connection $\nabla$ on a manifold M we stipulate a family of affine connections $\nabla^{A}$ of the form

$$
\nabla_{X}^{A} Y \doteq \nabla_{X} Y+A(X, Y)
$$

in which $X, Y$ are vector fields and $A$ a tensor field of covariant rank 2 and contravariant rank 1.
a. Show that $\nabla^{A}$ is indeed an affine connection for any choice of $A$.

Let $f, f_{1}, f_{2} \in C^{\infty}(\mathrm{M}), X, X_{1}, X_{2}, Y, Y_{1}, Y_{2} \in \mathrm{TM}$, and $\lambda, \lambda_{1}, \lambda_{2} \in \mathbb{R}$, then, by definition, an affine connection must satisfy

1. $\nabla_{f_{1} X_{1}+f_{2} X_{2}} Y=f_{1} \nabla_{X_{1}} Y+f_{2} \nabla_{X_{2}} Y$,
2. $\nabla_{X}\left(\lambda_{1} Y_{1}+\lambda_{2} Y_{2}\right)=\lambda_{1} \nabla_{X} Y_{1}+\lambda_{2} \nabla_{X} Y_{2}$,
3. $\nabla_{X}(f Y)=f \nabla_{X} Y+X f Y$.

Assuming this to be the case for $\nabla$, we must work out the left hand sides for $\nabla^{A}$ and verify that we obtain the same right hand sides with $\nabla$ likewise replaced by $\nabla^{A}$. Since $A$ is bilinear, we may use

1. $A\left(f_{1} X_{1}+f_{2} X_{2}, Y\right)=f_{1} A\left(X_{1}, Y\right)+f_{2} A\left(X_{2}, Y\right)$,
2. $A\left(X, \lambda_{1} Y_{1}+\lambda_{2} Y_{2}\right)=\lambda_{1} A\left(X, Y_{1}\right)+\lambda_{2} A\left(X, Y_{2}\right)$,
3. $A(X, f Y)=f A(X, Y)$.

Indeed, this implies

1. $\nabla_{f_{1} X_{1}+f_{2} X_{2}}^{A} Y=\nabla_{f_{1} X_{1}+f_{2} X_{2}} Y+A\left(f_{1} X_{1}+f_{2} X_{2}, Y\right)=f_{1} \nabla_{X_{1}} Y+f_{2} \nabla_{X_{2}} Y+f_{1} A\left(X_{1}, Y\right)+f_{2} A\left(X_{2}, Y\right)=$ $f_{1} \nabla_{X_{1}}^{A} Y+f_{2} \nabla_{X_{2}}^{A} Y$,
2. $\nabla_{X}^{A}\left(\lambda_{1} Y_{1}+\lambda_{2} Y_{2}\right)=\nabla_{X}\left(\lambda_{1} Y_{1}+\lambda_{2} Y_{2}\right)+A\left(X, \lambda_{1} Y_{1}+\lambda_{2} Y_{2}\right)=\lambda_{1} \nabla_{X} Y_{1}+\lambda_{2} \nabla_{X} Y_{2}+\lambda_{1} A\left(X, Y_{1}\right)+\lambda_{2} A\left(X, Y_{2}\right)=$ $\lambda_{1} \nabla_{X}^{A} Y_{1}+\lambda_{2} \nabla_{X}^{A} Y_{2}$,
3. $\nabla_{X}^{A}(f Y)=\nabla_{X}(f Y)+A(X, f Y)=f \nabla_{X} Y+X f Y+f A(X, Y)=f \nabla_{X}^{A} Y+X f Y$.

Thus $\nabla^{A}$ is an affine connection if $\nabla$ is.
Definition. The torsion $T$ of an affine connection $\nabla$ is defined by $T(X, Y) \doteq \nabla_{X} Y-\nabla_{Y} X-[X, Y]$.
b. Prove that $T$ is a tensor field by showing that the following properties hold for all scalar fields $f_{1,2} \in C^{\infty}(\mathbf{M})$ as well as all vector fields $X, Y, X_{1,2}, Y_{1,2}$ :
b1. $T\left(f_{1} X_{1}+f_{2} X_{2}, Y\right)=f_{1} T\left(X_{1}, Y\right)+f_{2} T\left(X_{2}, Y\right)$, and
b2. $T\left(X, f_{1} Y_{1}+f_{2} Y_{2}\right)=f_{1} T\left(X, Y_{1}\right)+f_{2} T\left(X, Y_{2}\right)$.

Note that b 2 follows from b 1 by the observation that $T$ is antisymmetric, since

$$
T\left(X, f_{1} Y_{1}+f_{2} Y_{2}\right)=-T\left(f_{1} Y_{1}+f_{2} Y_{2}, X\right)=-\left(f_{1} T\left(Y_{1}, X\right)+f_{2} T\left(Y_{2}, X\right)\right)=f_{1} T\left(X, Y_{1}\right)+f_{2} T\left(X, Y_{2}\right)
$$

Thus we only need to prove b1. The proof relies on the defining properties of the affine connection $\nabla$ and the commutator [, ], viz.

$$
\begin{aligned}
& T\left(f_{1} X_{1}+f_{2} X_{2}, Y\right)=\nabla_{f_{1} X_{1}+f_{2} X_{2} Y}-\nabla_{Y}\left(f_{1} X_{1}+f_{2} X_{2}\right)-\left[f_{1} X_{1}+f_{2} X_{2}, Y\right] \stackrel{*}{=} \\
& \quad f_{1} \nabla_{X_{1}} Y+f_{2} \nabla_{X_{2}} Y-f_{1} \nabla_{Y} X_{1}-f_{2} \nabla_{Y} X_{2}-f_{1}\left[X_{1}, Y\right]-f_{2}\left[X_{2}, Y\right]=f_{1} T\left(X_{1}, Y\right)+f_{2} T\left(X_{2}, Y\right) .
\end{aligned}
$$

Note that in the identity marked by $*$ all terms have cancelled that involve the action of the vector field (or derivative) $Y$ on the scalar functions $f_{1}, f_{2}$. These terms arise both from the affine connection term involving $\nabla_{Y}$ (notably via defining property 3 ) as well as from the part of the commutator term in which $Y$ acts as the outermost vector field.
c. Show that the torsion $T^{A}$ of $\nabla^{A}$ is $A$-invariant if and only if $A$ is symmetric.

We have
$T^{A}(X, Y) \doteq \nabla_{X}^{A} Y-\nabla_{Y}^{A} X-[X, Y]=\nabla_{X} Y+A(X, Y)-\nabla_{Y} X-A(Y, X)-[X, Y] \stackrel{\star}{=} \nabla_{X} Y-\nabla_{Y} X-[X, Y] \doteq T(X, Y)$, iff $A(X, Y)=A(Y, X)$, which has been used in $*$.
d. Suppose $\nabla$ has nonzero torsion $T$. Find $A$ such that $\nabla^{A}$ is torsion-free, i.e. $T^{A}=0$.

From c it is clear that we may restrict our attention to antisymmetric tensors (w.r.t. their vector arguments). Take $A(X, Y)=-\frac{1}{2} T(X, Y)$. This results in $\nabla_{X}^{A} Y=\nabla_{X} Y+A(X, Y) \doteq \nabla_{X} Y-\frac{1}{2} T(X, Y)=\frac{1}{2} \nabla_{X} Y+\frac{1}{2} \nabla_{Y} X-\frac{1}{2}[X, Y]$, with torsion $T^{A}(X, Y) \doteq$ $\nabla_{X}^{A} Y-\nabla_{Y}^{A} X-[X, Y] \doteq\left(\frac{1}{2} \nabla_{X} Y+\frac{1}{2} \nabla_{Y} X-\frac{1}{2}[X, Y]\right)-\left(\frac{1}{2} \nabla_{Y} X+\frac{1}{2} \nabla_{X} Y-\frac{1}{2}[Y, X]\right)-[X, Y]=0$. In the last step we have used antisymmetry of the commutator, $[Y, X]=-[X, Y]$.
$\Gamma$-Symbols. The $\Gamma$-symbols of $\nabla$ relative to a coordinate basis $\left\{\partial_{i}\right\}$ are defined by $\nabla_{\partial_{j}} \partial_{i}=\Gamma_{i j}^{k} \partial_{k}$.
e. Show that $\nabla$ is torsion-free (cf. problem d) if and only if $\Gamma_{j i}^{k}=\Gamma_{i j}^{k}$.

Consider the coordinate vector fields $X=\partial_{i}$ and $Y=\partial_{j}$. From the definition of torsion-freeness we infer that

$$
0=\nabla_{\partial_{i}} \partial_{j}-\nabla_{\partial_{j}} \partial_{i}-\left[\partial_{i}, \partial_{j}\right]=\nabla_{\partial_{i}} \partial_{j}-\nabla_{\partial_{j}} \partial_{i},
$$

since partial derivatives commute, whence $\nabla_{\partial_{i}} \partial_{j}=\nabla_{\partial_{j}} \partial_{i}$. Using the definition of the $\Gamma$-symbols we thus find that

$$
\Gamma_{j i}^{k} \doteq\left\langle d x^{k}, \nabla_{\partial_{i}} \partial_{j}\right\rangle=\left\langle d x^{k}, \nabla_{\partial_{j}} \partial_{i}\right\rangle \doteq \Gamma_{i j}^{k} .
$$

f. Show that if $\nabla$ is not torsion-free, then the connection symbols for the torsion-free connection $\nabla^{A}$ constructed in problem d are given by $\bar{\Gamma}_{i j}^{k}=\frac{1}{2}\left(\Gamma_{i j}^{k}+\Gamma_{j i}^{k}\right)$.

We have, by definition,

$$
\bar{\Gamma}_{i j}^{k} \doteq\left\langle d x^{k}, \nabla_{\partial_{j}}^{A} \partial_{i}\right\rangle \doteq\left\langle d x^{k}, \nabla_{\partial_{j}} \partial_{i}+A\left(\partial_{j}, \partial_{i}\right)\right\rangle \stackrel{\mathrm{d}}{=}\left\langle d x^{k}, \frac{1}{2} \nabla_{\partial_{j}} \partial_{i}+\frac{1}{2} \nabla_{\partial_{i}} \partial_{j}-\frac{1}{2}\left[\partial_{j}, \partial_{i}\right]\right\rangle .
$$

Again the commutator of partial derivatives vanishes identically, and so by linearity w.r.t. the second argument of the Kronecker tensor we obtain

$$
\bar{\Gamma}_{i j}^{k}=\frac{1}{2}\left\langle d x^{k}, \nabla_{\partial_{j}} \partial_{i}\right\rangle+\frac{1}{2}\left\langle d x^{k}, \nabla_{\partial_{i}} \partial_{j}\right\rangle \doteq \frac{1}{2} \Gamma_{i j}^{k}+\frac{1}{2} \Gamma_{j i}^{k} .
$$

## 3. Einstein Field Equations: Dark Energy

Definition. The holor of the Riemann tensor is defined as

$$
R_{\mu \sigma \nu}^{\rho}=\left\langle d x^{\rho}, \mathscr{R}\left(\partial_{\sigma}, \partial_{\nu}\right) \partial_{\mu}\right\rangle=\partial_{\sigma} \Gamma_{\mu \nu}^{\rho}-\partial_{\nu} \Gamma_{\mu \sigma}^{\rho}+\Gamma_{\lambda \sigma}^{\rho} \Gamma_{\mu \nu}^{\lambda}-\Gamma_{\lambda \nu}^{\rho} \Gamma_{\mu \sigma}^{\lambda} .
$$

The holor of the covariant Riemann tensor is defined as $R_{\lambda \mu \sigma \nu}=g_{\lambda \rho} R^{\rho}{ }_{\mu \sigma \nu}$. The Ricci tensor has holor $R_{\mu \nu}=R^{\rho}{ }_{\mu \rho \nu}$. The Ricci scalar equals $R=g^{\mu \nu} R_{\mu \nu}$.

The Einstein field equations in $(3+1)$-dimensional spacetime comprise a system of nonlinear partial differential equations for the Lorentzian-type metric tensor $g_{\mu \nu}$, driven by the distribution of matter and energy captured by the stress-energy tensor $T_{\mu \nu}$ :

$$
R_{\mu \nu}-\frac{1}{2} R g_{\mu \nu}+\Lambda g_{\mu \nu}=\frac{8 \pi G}{c^{4}} T_{\mu \nu}
$$

Here $\Lambda, G$ and $c$ denote cosmological constant, Newton's gravitational constant, and speed of light in vacuum, respectively, and $R_{\mu \nu}$ and $R$ are the Ricci tensor and Ricci scalar, to be regarded as nonlinear functions of the metric tensor and its derivatives. The signature ${ }^{1}$ of $g_{\mu \nu}$ is taken as $(-+++)$.

In Einstein's original formulation the cosmological term $\Lambda g_{\mu \nu}$ was absent. Einstein included this term later so as to allow for a non-expanding and non-contracting universe.

In the absence of matter and energy, $T_{\mu \nu}=0$, we refer to $(\star)$ as the vacuum field equations, and to any solution $g_{\mu \nu}$ as a vacuum solution.
a1. Show that $\Gamma_{\sigma \rho}^{\rho}=\partial_{\sigma} \ln \sqrt{|g|}$.
[Hint: Use $D_{\sigma} g_{\mu \nu}=0$ and the definition of a cofactor matrix and its relation to matrix inverse.]

From $D_{\sigma} g_{\mu \nu}=0$ (metric compatibility) we obtain $\partial_{\sigma} g_{\mu \nu}=\Gamma_{\mu \sigma}^{\rho} g_{\rho \nu}+\Gamma_{\nu \sigma}^{\rho} g_{\mu \rho}$. Therefore, using the chain rule and the definition of the cofactor matrix, $\partial_{\sigma} g=\tilde{g}^{\mu \nu} \partial_{\sigma} g_{\mu \nu}=g g^{\mu \nu} \partial_{\sigma} g_{\mu \nu}$. Inserting the previous expression for the metric derivatives in terms of the $\Gamma$-symbols then yields $\partial_{\sigma} g=2 g \Gamma_{\sigma \rho}^{\rho}$, from which the result follows by elementary calculus.
a2. Show that the Ricci tensor is symmetric.
$R_{\mu \nu}=R^{\rho}{ }_{\mu \rho \nu}=\partial_{\rho} \Gamma_{\mu \nu}^{\rho}-\partial_{\nu} \Gamma_{\mu \rho}^{\rho}+\Gamma_{\lambda \rho}^{\rho} \Gamma_{\mu \nu}^{\lambda}-\Gamma_{\lambda \nu}^{\rho} \Gamma_{\mu \rho}^{\lambda}$. All terms on the r.h.s. are obviously symmetric w.r.t. $\mu$ and $\nu$, except for the second one. A closer inspection, however, notably a1, reveals that $\partial_{\nu} \Gamma_{\mu \rho}^{\rho}=\partial_{\mu \nu} \ln \sqrt{|g|}$, which is clearly also symmetric.
b1. Show that the vacuum field equations reduce to $R_{\mu \nu}-\Lambda g_{\mu \nu}=0$.

Taking the trace of $(\star)$ by contraction with $g^{\mu \nu}$ yields, with the help of $g^{\mu \nu} g_{\mu \nu}=4$, the identity $R-4 \Lambda=0$, allowing elimination of $R$ from the vacuum field equations, from which the result then readily follows. Note the sign switch in the cosmological term.
b2. Show that ( $\star$ ) is equivalent to $R_{\mu \nu}-\Lambda g_{\mu \nu}=\frac{8 \pi G}{c^{4}}\left(T_{\mu \nu}-\frac{1}{2} T g_{\mu \nu}\right)$, and explain the scalar field $T$.
As before, take the trace of $(\star)$ by contraction with $g^{\mu \nu}$, from which it follows that $R=4 \Lambda-\frac{8 \pi G}{c^{4}} T$, in which $T=g^{\mu \nu} T_{\mu \nu}$. Replace $R$ on the 1.h.s. of $(\star)$ by this expression and the result follows. Note that al is a special case consistent with this result.

The cosmological term in ( $\star$ ) may be interpreted as a mysterious "vacuum" or "dark energy", viz. by replacing $T_{\mu \nu}$ on the r.h.s. by $T_{\mu \nu}^{\Lambda} \doteq T_{\mu \nu}+T_{\mu \nu}^{\mathrm{vac}}$ in return for dropping the geometric $\Lambda$-term on the 1.h.s.
c. What should we take for the vacuum stress-energy tensor $T_{\mu \nu}^{\mathrm{vac}}$ in this case?

A straightforward algebraic manipulation gives $T_{\mu \nu}^{\mathrm{vac}}=-\frac{\Lambda c^{4}}{8 \pi G} g_{\mu \nu}$.
Lemma. The holor of the covariant Riemann tensor has the following symmetry properties:

[^0]－$R_{\lambda \sigma \mu \nu}=-R_{\sigma \lambda \mu \nu}=-R_{\lambda \sigma \nu \mu}$
－$R_{\lambda \sigma \mu \nu}=R_{\mu \nu \lambda \sigma}$
－$R_{\lambda \sigma \mu \nu}+R_{\lambda \mu \nu \sigma}+R_{\lambda \nu \sigma \mu}=0$（first Bianchi identity）
－$D_{\rho} R_{\lambda \sigma \mu \nu}+D_{\mu} R_{\lambda \sigma \nu \rho}+D_{\nu} R_{\lambda \sigma \rho \mu}=0$（second Bianchi identity）

Note that the Bianchi identities can be condensed to $R_{\lambda[\sigma \mu \nu]}=0$ and $R_{\lambda \sigma[\mu \nu ; \rho]}=0$ ，respectively．
We stipulate a metric that satisfies $R_{\rho \sigma \mu \nu}=k\left(g_{\rho \mu} g_{\sigma \nu}-g_{\rho \nu} g_{\sigma \mu}\right)$ for some constant $k \in \mathbb{R}$ ．
d．Show that，if such a metric exists，then it is a solution to the vacuum equations for suitably chosen $k$ ．

Computing the Ricci tensor and Ricci scalar yields $R_{\mu \nu}=3 k g_{\mu \nu}$ ，respectively $R=12 k$ ．This is consistent with the vacuum field equations， cf．problem b1，provided we set $k=\Lambda / 3$ ．

Conservation of energy and momentum can be geometrically formulated as $D_{\mu} T^{\mu \nu}=0$ ．
e．Show that Einstein＇s field equations（ $\star$ ）are consistent with energy－momentum conservation．
We have to show that $D_{\mu}\left(R^{\mu \nu}-\frac{1}{2} R g^{\mu \nu}+\Lambda g^{\mu \nu}\right)=0$ ，i．e．

$$
(\dagger) \quad D_{\mu} R^{\mu \nu}=\frac{1}{2} g^{\mu \nu} D_{\mu} R
$$

The proof relies on the second Bianchi identity，combined with covariant constancy of the（dual）metric，$D_{\rho} g^{\mu \nu}=0$（which admits raising and lowering indices after a covariant derivative），and symmetries of the Riemann and Ricci tensors（stated in the lemma，respectively proven in a2）．Namely，contract the second Bianchi identity in the form as stated in the lemma with $g^{\lambda \mu} g^{\sigma \nu}$ and work out the result．This yields

$$
D_{\rho} R-2 g^{\lambda \mu} D_{\mu} R_{\rho \lambda}=0
$$

which is equivalent to $(\dagger)$ and completes the proof．
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By testing you remotely in this fashion，we express our trust that you will adhere to the ethical standard of behaviour expected of you．This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability，without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test．

Text to be copied（with optional remarks）：
$\mathrm{K}_{\mathrm{O}}$ I made this test to the best of my own ability，without seeking or accepting the help of any source not
explicitly allowed by the conditions of the test．
Remarks from the student：


[^0]:    ${ }^{1}$ This convention is used in C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, W. H. Freeman, 1973.

