EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Friday April 11, 2014. Time: 09h00–12h00. Place: AUD 15.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, or other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.

GOOD LUCK!

(**35**) **1.** VECTOR SPACES

We consider the linear space V over $\mathbb{R}$ consisting of all infinite sequences $s=(s_1, s_2, s_3, \ldots)\in\mathbb{R}^\infty$, furnished with the usual definitions of vector addition and scalar multiplication. You may take it for granted that V is indeed a linear space.

(5) **a.** Explain what is meant by "the usual definitions of vector addition and scalar multiplication".

If $s=(s_1, s_2, s_3, \ldots), t=(t_1, t_2, t_3, \ldots)\in\mathbb{R}^\infty$ and $\lambda, \mu \in \mathbb{R}$, then $\lambda s + \mu t=(\lambda s_1 + \mu t_1, \lambda s_2 + \mu t_2, \lambda s_3 + \mu t_3, \ldots)$.

The subset $W \subset V$ is defined as the set of converging sequences:

$$W = \{s \in V \mid -\infty < \lim_{n\to\infty} s_n < \infty\}$$

(10) **b.** Show that W is itself a linear space over $\mathbb{R}$.

Since $W \subset V$, with V a linear space, it suffices to prove closure of W by the subspace theorem. Let $s=(s_1, s_2, s_3, \ldots)\in W$ and $t = (t_1, t_2, t_3, \ldots) \in W$, with limts $\lim_{n\to\infty} s_n = \sigma$ and $\lim_{n\to\infty} t_n = \tau$ for some $\sigma, \tau \in \mathbb{R}$, say. Then the sequence $\ell=\lambda s + \mu t$, given by $(\ell_1, \ell_2, \ell_3, \ldots)=(\lambda s_1 + \mu t_1, \lambda s_2 + \mu t_2, \lambda s_3 + \mu t_3, \ldots)$, converges, viz. $\lim_{n\to\infty} \ell_n=\lambda\sigma + \mu\tau \in \mathbb{R}$.

We subsequently consider the subsets $W_a \subset W$ for each fixed $a \in \mathbb{R}$, defined as follows:

$$W_a = \{s \in W \mid \lim_{n\to\infty} s_n = a\}$$

(10) **c.** Does W_a define a linear space? Prove your statement.

For nonzero $a \in \mathbb{R}$, $W_a \subset W$ is not closed and therefore not a linear space, since if $s = (s_1, s_2, s_3, \ldots) \in W_a$ and $t = (t_1, t_2, t_3, \ldots) \in W_a$, then $\lim_{n\to\infty}(\lambda s + \mu t)_n = \lambda \lim_{n\to\infty} s_n + \mu \lim_{n\to\infty} t_n = (\lambda + \mu)a$, so $\lambda s + \mu t \in W_{(\lambda+\mu)a}$ for all $\lambda, \mu \in \mathbb{R}$. For closure we must require that $(\lambda + \mu)a = a$ for all $\lambda, \mu \in \mathbb{R}$, which holds if and only if $a=0$. By the subspace theorem $W_0 \subset W$ does indeed define a linear space.

Let $C \subset V$ be the set of infinite sequences with converging partial sums, i.e.

$$C = \{s \in V \mid -\infty < \sum_{n=1}^{\infty} s_n \stackrel{\text{def}}{=} \lim_{N\to\infty} \sum_{n=1}^{N} s_n < \infty\}$$

(10) **d.** Show that C is a linear space over $\mathbb{R}$.
(*Hint:* Recall c and argue why the subspace theorem applies.)

Terms in a converging series constitute a sequence that converges to zero, i.e. $C \subset W_0$. Consequently, since W_0 is a linear space, we may use the subspace theorem. If $s = (s_1, s_2, s_3, \ldots) \in C$ and $t = (t_1, t_2, t_3, \ldots) \in C$, then we have $\lambda s + \mu t = (\lambda s_1 + \mu t_1, \lambda s_2 + \mu t_2, \lambda s_3 + \mu t_3, \ldots)$ for all $\lambda, \mu \in \mathbb{R}$. That this is an element of C follows from the fact that $\lim_{N\to\infty} \sum_{n=1}^{N} (\lambda s + \mu t)_n = \lim_{N\to\infty} \sum_{n=1}^{N} (\lambda s_n + \mu t_n) = \lambda \lim_{N\to\infty} \sum_{n=1}^{N} s_n + \mu \lim_{N\to\infty} \sum_{n=1}^{N} t_n$ is finite for all $\lambda, \mu \in \mathbb{R}$.

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(**20**) **2.** Inner Product (Exam June 28, 2006, Problem 1)

In this problem we consider a vector space V over the scalar field $\mathbb{R}$, equipped with a real-valued inner product, $\langle\ |\ \rangle : V \times V \to \mathbb{R} : (v, w) \mapsto \langle v|w\rangle$. We henceforth refer to a real-valued inner product simply as "inner product".

Lemma. For each pair of vectors $v, w \in V$ the following *Schwartz inequality* holds:

$$|\langle v|w\rangle| \leq \sqrt{\langle v|v\rangle \, \langle w|w\rangle}\,.$$

(5) **a.** Prove this lemma, exploiting the defining properties of the inner product.
(*Hint:* Consider the trivial inequality $\langle \lambda v + w|\lambda v + w\rangle \geq 0$ for given $v, w \in V$ and arbitrary $\lambda \in \mathbb{R}$. Why, by the way, is this inequality "trivial"?)

The inequality $\langle \lambda v + w|\lambda v + w\rangle \geq 0$ holds trivially, because by definition an inner product is non-negative definite on V. Using bilinearity and symmetry we may rewrite the inequality as

$$\langle v|v\rangle\lambda^2 + 2\langle v|w\rangle\lambda + \langle w|w\rangle \geq 0\,.$$

The left hand side is apparently a non-negative quadratic function in $\lambda \in \mathbb{R}$ (the corresponding graph is a parabola pointing downward and touching the λ-axis in at most one point). The discriminant thus has to be non-positive:

$$4\langle v|w\rangle^2 - 4\langle v|v\rangle\langle w|w\rangle \leq 0\,.$$

This immediately yields the Schwartz inequality.

Theorem. Every inner product $\langle\ |\ \rangle : V \times V \to \mathbb{R}$ induces a norm, $\|\ \| : V \to \mathbb{R}$, as follows:

$$\|v\| \stackrel{\text{def}}{=} \sqrt{\langle v|v\rangle}\,.$$

This norm is referrred to as the *norm induced by the inner product.*

(5) **b.** Prove this theorem, using the defining properties of the inner product.

1. For arbitrary $v \in V$ we have $\|v\| \stackrel{\text{def}}{=} \sqrt{\langle v|v\rangle} \geq 0$, in which the inequality follows from the fact that any inner product satisfies $\langle v|v\rangle \geq 0$. Moreover, equality holds if and only if $v = 0 \in V$ as a consequence of non-degeneracy of the inner product.

2. For arbitrary $v \in V$ en $\lambda \in \mathbb{R}$ we have $\|\lambda v\| \stackrel{\text{def}}{=} \sqrt{\langle \lambda v | \lambda v \rangle} \stackrel{*}{=} \sqrt{\lambda^2 \langle v | v \rangle} = |\lambda| \sqrt{\langle v | v \rangle} \stackrel{\text{def}}{=} |\lambda| \|v\|$. The identity $*$ follows from bilinearity of the inner product.

3. For all $v, w \in V$ we have $\|v+w\|^2 \stackrel{\text{def}}{=} \langle v+w | v+w \rangle \stackrel{*}{=} \langle v|v \rangle + 2\langle v|w \rangle + \langle w|w \rangle \stackrel{\star}{\leq} \langle v|v \rangle + 2\sqrt{\langle v|v \rangle \langle w|w \rangle} + \langle w|w \rangle \stackrel{\text{def}}{=} \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2$. Therefore it follows that $\|v+w\| \leq \|v\| + \|w\|$. In $*$ we have made use of bilinearity and symmetry of the inner product, and in $\star$ of the Schwartz inequality.

(5) **c.** Prove that for all $v, w \in V$ we have

$$\frac{1}{4}\|v+w\|^2 - \frac{1}{4}\|v-w\|^2 = \langle v|w \rangle \,.$$

Using the definition of the norm induced by the inner product and of bilinearity and symmetry of the inner product, we may rewrite the left hand side as follows:

$$\frac{1}{4}\|v+w\|^2 - \frac{1}{4}\|v-w\|^2 \stackrel{\text{def}}{=} \frac{1}{4}\langle v+w|v+w \rangle - \frac{1}{4}\langle v-w|v-w \rangle = \frac{1}{4}\left(\langle v|v \rangle + 2\langle v|w \rangle + \langle w|w \rangle\right) - \frac{1}{4}\left(\langle v|v \rangle - 2\langle v|w \rangle + \langle w|w \rangle\right) = \langle v|w \rangle \,.$$

(5) **d.** Prove that for all $v, w \in V$ we have

$$\frac{1}{2}\|v+w\|^2 + \frac{1}{2}\|v-w\|^2 = \|v\|^2 + \|w\|^2 \,.$$

In analogy with the previous problem we rewrite terms on the left hand side as follows:

$$\|v \pm w\|^2 \stackrel{\text{def}}{=} \langle v \pm w | v \pm w \rangle = \langle v|v \rangle \pm 2\langle v|w \rangle + \langle w|w \rangle \,.$$

Taking the average of the "+" and "−" terms, the mixed terms cancel:

$$\frac{1}{2}\|v+w\|^2 + \frac{1}{2}\|v-w\|^2 = \langle v|v \rangle + \langle w|w \rangle \stackrel{\text{def}}{=} \|v\|^2 + \|w\|^2 \,.$$

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(25) 3. ALGEBRAS

A so-called *octonion*, or *Cayley's number*, can be written as a real linear combination of eight "unit octonions", e_0, e_1, e_2, e_3, e_4, e_5, e_6 and e_7, say. Together, these linear combinations constitute the set

$$\mathbb{O} = \left\{ x = \sum_{i=0}^{7} x_i e_i \mid x_i \in \mathbb{R} \right\}$$

We conjecture that $\mathbb{O}$ forms an 8-dimensional real vector space.

(10) **a.** Show that if $x = \sum_{i=0}^{7} x_i e_i \in \mathbb{O}$, $y = \sum_{i=0}^{7} y_i e_i \in \mathbb{O}$ for $x_i, y_i \in \mathbb{R}$, and $\lambda \in \mathbb{R}$, then $z = x + \lambda y \in \mathbb{O}$.

We have $z = x + \lambda y = \sum_{i=0}^{7} x_i e_i + \lambda \sum_{i=0}^{7} y_i e_i = \sum_{i=0}^{7} (x_i + \lambda y_i)\, e_i \in \mathbb{O}$.

In an attempt to turn the vector space $\mathbb{O}$ into an algebra we introduce multiplication. Its defining rules can be deduced from *Fanos' plane* (see Fig. 1). The following rules apply:

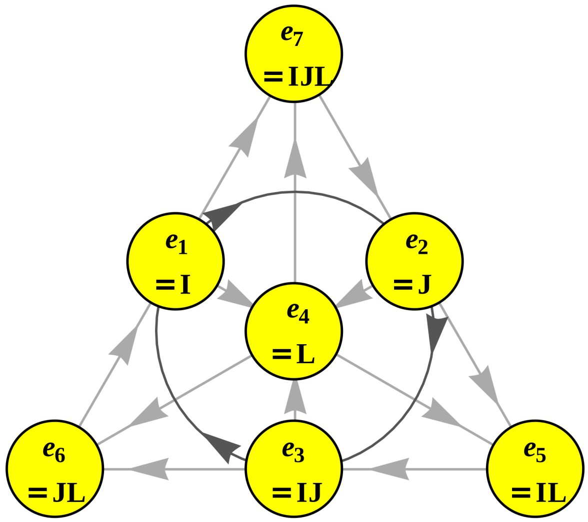

Figure 1: Fano's plane.

- each of the seven nodes in the diagram represents a unit octonion as indicated (e_0 is not depicted in the diagram);
- if (a, b, c) is an ordered triple of unit octonions lying on a given line with the order specified by the direction of the arrow, then $ab = c$ and $ba = -c$, together with cyclic permutations (thus e.g. $e_1 e_2 = e_3$, $e_1 e_6 = -e_7$, et cetera);
- e_0 is the multiplicative identity element (often written as "1");
- $e_i e_i = -e_0$ for each $i = 1, \ldots, 7$.

With the usual distributive laws for products of linear combinations this completely defines the multiplicative structure of $\mathbb{O}$. Example:

$$(3e_0 + e_1)(2e_2 - e_6) = 6e_0e_2 - 3e_0e_6 + 2e_1e_2 - e_1e_6 = 6e_2 + 2e_3 - 3e_6 + e_7 \,.$$

(10) **b.** Complete the multiplication table in Fig. 2 (see appendix) with the help of Fano's plane and the stated rules.
(*Attention:* Do not forget to hand this in with your name and student ID on it.)

See Fig. 3 in the appendix.

The *associator* $[a, b, c]$ of three octonions $a, b, c \in \mathbb{O}$ is given by

$$[a, b, c] = (ab)c - a(bc) \,.$$

Recall that, according to our definition, associativity is one of the basic axioms of an algebra. A linear space endowed with a multiplicative structure that fulfills this axiom is also referred to as an *associative algebra.*

(5) **c.** Show that $\mathbb{O}$ is not an associative algebra.

We need to find an example of a nonvanishing associator. There are multiple options. One example is $[e_1, e_2, e_4] = (e_1e_2)e_4 - e_1(e_2e_4) = e_3e_4 - e_1e_6 = e_7 + e_7 = 2e_7 \neq 0 \in \mathbb{O}$, in which we have used Fig. 3.

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(**20**) **4.** Distribution Theory & Fourier Analysis

The Fourier transform $\widehat{T} \in \mathscr{S}'(\mathbb{R}^n)$ of a distribution $T \in \mathscr{S}'(\mathbb{R}^n)$ is defined as follows:

$$\widehat{T}(\phi) = T(\widehat{\phi}) \quad \text{for all test functions } \phi \in \mathscr{S}(\mathbb{R}^n), \qquad (\star)$$

in which

$$\widehat{\phi}(\omega) = \int_{\mathbb{R}^n} e^{-i\omega\cdot x}\, \phi(x)\, dx\,.$$

The purpose of this problem is to motivate this definition.

To this end, consider any function $f : \mathbb{R}^n \to \mathbb{C}$ of polynomial growth for which the Fourier integral

$$\widehat{f}(\omega) = \int_{\mathbb{R}^n} e^{-i\omega\cdot x}\, f(x)\, dx$$

is well-defined and yields a function $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ of polynomial growth. Denote by $T_f \in \mathscr{S}'(\mathbb{R}^n)$ the corresponding *regular* tempered distribution, i.e.

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\, \phi(x)\, dx$$

for all test functions $\phi \in \mathscr{S}(\mathbb{R}^n)$. It is then natural to define $\widehat{T}_f \stackrel{\text{def}}{=} T_{\widehat{f}}$, i.e.

$$\widehat{T}_f(\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \widehat{f}(\xi)\, \phi(\xi)\, d\xi\,. \qquad (*)$$

(10) **a.** Show that this definition implies $\widehat{T}_f(\phi) = T_f(\widehat{\phi})$ for any $\phi \in \mathscr{S}(\mathbb{R}^n)$.
(*Hint:* In $(*)$ apply the Fourier reconstruction formula in the following form: $\phi(\xi) = \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} e^{i\xi\cdot x}\, \widehat{\phi}(x)\, dx$.)

Following the hint (in $*$) we write

$$\widehat{T}_f(\phi) \stackrel{\text{def}}{=} T_{\widehat{f}}(\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \widehat{f}(\xi)\, \phi(\xi)\, d\xi \stackrel{*}{=} \int_{\mathbb{R}^n} \widehat{f}(\xi) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot x}\, \widehat{\phi}(x)\, dx \right) d\xi\,.$$

Interchanging the order of integration this is seen to be equivalent to

$$\int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{i\xi\cdot x}\, d\xi \right) \widehat{\phi}(x)\, dx \stackrel{\circ}{=} \int_{\mathbb{R}^n} f(x)\, \widehat{\phi}(x)\, dx = T_f(\widehat{\phi})\,.$$

In $\circ$ the Fourier reconstruction formula has been used once more, this time for the function f.

This result justifies the general definition ($\star$), even if $T \in \mathscr{S}'(\mathbb{R}^n)$ is not regular.

As an example, consider the (non-regular) Dirac point distribution $\delta \in \mathscr{S}'(\mathbb{R}^n)$, defined by $\delta(\phi) = \phi(0)$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

(10) **b.** Use the general definition ($\star$) to prove that $\widehat{\delta} = T_{\mathit{1}}$. Here $T_{\mathit{1}} \in \mathscr{S}'(\mathbb{R}^n)$ is the regular tempered distribution corresponding to the constant function $\mathit{1} : \mathbb{R}^n \to \mathbb{C} : x \mapsto \mathit{1}(x) = 1$.

Using the distributional definition of the Fourier transform (in $\star$) we find that

$$\widehat{\delta}(\phi) \stackrel{\star}{=} \delta(\widehat{\phi}) \stackrel{\text{def}}{=} \widehat{\phi}(0) \stackrel{\dagger}{=} \int_{\mathbb{R}^n} \phi(x)\, dx \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \mathit{1}(x)\, \phi(x)\, dx \stackrel{\text{def}}{=} T_{\mathit{1}}(\phi)\,.$$

Note that $\dagger$ uses a special case of the definition of the Fourier transform, viz.

$$\widehat{\phi}(\omega) = \int_{\mathbb{R}^n} e^{-i\omega\cdot x} \phi(x)\, dx\,,$$

for $\omega = 0 \in \mathbb{R}^n$.

THE END

Name: **Student ID:**

x	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0							
e_1		$-e_0$	e_3				$-e_7$	
e_2			$-e_0$					
e_3				$-e_0$				
e_4					$-e_0$			
e_5						$-e_0$		
e_6							$-e_0$	
e_7								$-e_0$

Figure 2: Multiplication table.

x	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Figure 3: Multiplication table completed.