EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Monday January 17, 2011. Time: 09h00–12h00. Place: AUD 4

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, or other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.

GOOD LUCK!

(**20**) **1.** Vector Space

We introduce the set $V = \mathbb{R}^2$ and furnish it with an addition and scalar multiplication operator, as follows. For all $(x, y) \in \mathbb{R}^2$, $(u, v) \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$ we define

$$(x, y) + (u, v) = (x + u, y + v) \quad \text{and} \quad \lambda \cdot (x, y) = (\lambda x, y)\,.$$

Show that, given these definitions, V does *not* constitute a vector space.

The only criterion this definition fails to pass is the distributivity requirement $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$ for any vector $v \in V$ and all $\lambda, \mu \in \mathbb{R}$. Indeed, we have in this particular case,

$$(\lambda + \mu) \cdot (x, y) \stackrel{\text{def}}{=} ((\lambda + \mu)x, y) \quad \text{whereas} \quad \lambda \cdot (x, y) + \mu \cdot (x, y) \stackrel{\text{def}}{=} (\lambda x, y) + (\mu x, y) \stackrel{\text{def}}{=} ((\lambda + \mu)x, 2y)\,.$$

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(**25**) **2.** Group Theory[1]

We define the following grey-value transformation: $T_\gamma : \mathbb{R} \to \mathbb{R} : s \mapsto T_\gamma(s) \stackrel{\text{def}}{=} e^{\gamma s}$, in which $\gamma \in \mathbb{R}$ is an arbitrary constant. We furnish the set of all transformations of this type, $G = \{T_\gamma \mid \gamma \in \mathbb{R}\}$, with an infix multiplication operator $\times$, as follows:

$$(T_\alpha \times T_\beta)(s) \stackrel{\text{def}}{=} T_\alpha(s)\, T_\beta(s) \quad \text{for all } s \in \mathbb{R}.$$

a. Prove that G constitutes a group. Proceed as follows:

(5) **a1.** Prove that G is closed with respect to multiplication, i.e. prove that $T_\alpha, T_\beta \in G$ implies $T_\alpha \times T_\beta \in G$ for all $\alpha, \beta \in \mathbb{R}$.

[1] Exam June 28, 2006, problem 3.

Suppose $T_\alpha(s) = e^{\alpha s}$, $T_\beta(s) = e^{\beta s}$ then $\left(T_\alpha \times T_\beta\right)(s) \stackrel{\text{def}}{=} T_\alpha(s)\, T_\beta(s) \stackrel{\text{def}}{=} e^{\alpha s} e^{\beta s} = e^{\alpha+\beta}(s) \stackrel{\text{def}}{=} T_{\alpha+\beta}(s)$ for all $s \in \mathbb{R}$, so $T_\alpha \times T_\beta = T_{\alpha+\beta} \in G$.

(5) **a2.** Prove that multiplication is associative on G, i.e. prove that $(T_\alpha \times T_\beta) \times T_\gamma = T_\alpha \times (T_\beta \times T_\gamma)$ for all $\alpha, \beta, \gamma \in \mathbb{R}$.

Using the proof of the previous part we obtain $(T_\alpha \times T_\beta) \times T_\gamma \stackrel{\text{a1}}{=} T_{\alpha+\beta} \times T_\gamma \stackrel{\text{a1}}{=} T_{(\alpha+\beta)+\gamma} = T_{\alpha+\beta+\gamma} = T_{\alpha+(\beta+\gamma)} \stackrel{\text{a1}}{=} T_\alpha \times T_{\beta+\gamma} \stackrel{\text{a1}}{=} T_\alpha \times (T_\beta \times T_\gamma)$ for all $\alpha, \beta, \gamma \in \mathbb{R}$.

(5) **a3.** Prove that G has a unit element, i.e. that there exists a $\nu \in \mathbb{R}$ such that $T_\nu \times T_\gamma = T_\gamma \times T_\nu = T_\gamma$ for all $\gamma \in \mathbb{R}$. Moreover, give the explicit value of $\nu \in \mathbb{R}$ corresponding to this unit element $T_\nu \in G$.

The unit element is $T_0 \in G$, for $T_\alpha \times T_0 \stackrel{\text{a1}}{=} T_{\alpha+0} = T_\alpha$ for all $\alpha \in \mathbb{R}$. Likewise we have $T_0 \times T_\alpha \stackrel{\text{a1}}{=} T_{0+\alpha} = T_\alpha$. A direct proof goes as follows: $T_0(s) \stackrel{\text{def}}{=} 1$ for all $s \in \mathbb{R}$, so $(T_\alpha \times T_0)(s) \stackrel{\text{def}}{=} T_\alpha(s) T_0(s) = e^{\alpha s} \cdot 1 = e^{\alpha s} = T_\alpha(s)$, and analogously, $(T_0 \times T_\alpha)(s) \stackrel{\text{def}}{=} T_0(s) T_\alpha(s) = 1 \cdot e^{\alpha s} = e^{\alpha s} = T_\alpha(s)$ for all $s \in \mathbb{R}$.

(5) **a4.** Finally prove that each element of G has an inverse, i.e. that for each $\eta \in \mathbb{R}$ there exists a $\theta \in \mathbb{R}$ such that $T_\eta \times T_\theta = T_\theta \times T_\eta = T_\nu$, in which $\nu \in \mathbb{R}$ denotes the parameter value corresponding to the unit element in part a3.

We already saw that $\nu = 0$. For arbitrary $\eta \in \mathbb{R}$ the inverse of T_η is given by $T_{-\eta} \in G$, since $T_\eta \times T_{-\eta} \stackrel{\text{a1}}{=} T_{\eta+(-\eta)} = T_0$, and likewise $T_{-\eta} \times T_\eta \stackrel{\text{a1}}{=} T_{(-\eta)+\eta} = T_0$.

(5) **b.** Is G commutative? If yes, prove, if no, provide a counterexample.

That G is indeed commutative has in fact already been proven under a1, for $T_\alpha \times T_\beta \stackrel{\text{a1}}{=} T_{\alpha+\beta} = T_{\beta+\alpha} = T_\beta \times T_\alpha$ for all $\alpha, \beta \in \mathbb{R}$.

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(25) 3. Distribution Theory

The parameterized function $f_a : \mathbb{R} \to \mathbb{R}$, with parameter $a > 0$, is defined as follows:

$$f_a(x) = \begin{cases} \dfrac{1}{a^2}(-|x| + a) & \text{if } x \in [-a, a] \\ 0 & \text{elsewhere} \end{cases}$$

(5) **a.** Sketch the graph of $y = f_a(x)$ in the xy-plane, and compute the area enclosed by this graph and the x-axis.

See Figure 1. The area under the graph invariably equals $2 \times \frac{1}{2}\, a \times \frac{1}{a} = 1$ irrespective of the value of $a > 0$.

The regular tempered distribution $T_{f_a} : \mathscr{S}(\mathbb{R}) \to \mathbb{R}$ associated with the function f_a is given by

$$T_{f_a}(\phi) = \int_{-\infty}^{\infty} f_a(x)\, \phi(x)\, dx$$

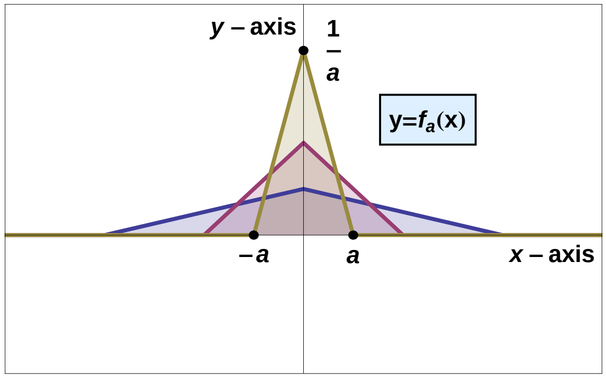

Figure 1: Graphs of $y = f_a(x)$ for three distinct values of a. The area under a graph invariably equals 1.

for any smooth test function $\phi \in \mathscr{S}(\mathbb{R})$.

(10) **b.** Show that $T_{f_a}(\phi) = \frac{1}{a}\int_{-a}^{a} \phi(x)\,dx + \frac{1}{a^2}\int_{-a}^{0} x\,\phi(x)\,dx - \frac{1}{a^2}\int_{0}^{a} x\,\phi(x)\,dx.$

$$T_{f_a}(\phi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f_a(x)\,\phi(x)\,dx \stackrel{\text{def}}{=} \frac{1}{a^2}\int_{-a}^{a}(-|x|+a)\,\phi(x)\,dx = \frac{1}{a}\int_{-a}^{a}\phi(x)\,dx + \frac{1}{a^2}\int_{-a}^{0} x\,\phi(x)\,dx - \frac{1}{a^2}\int_{0}^{a} x\,\phi(x)\,dx.$$

We now consider the limit of vanishing parameter $a \downarrow 0$. It is clear that the function f_a is ill-defined in this limit. We wish to investigate whether the regular tempered distribution T_{f_a} does have a well-defined limit. To this end we recall Taylor's theorem, which allows us to use the following second order expansion for the test function around the origin:

$$\phi(x) = \phi(0) + \phi'(0)\,x + \frac{1}{2}\phi''(\xi(x))\,x^2\,, \qquad (*)$$

for any $x \in (-a, a)$ and some $\xi(x)$ in-between x and 0. The last term on the right hand side is referred to as the Lagrange remainder, and is sometimes simplified as $\mathcal{O}(x^2)$.

Finally, recall the Dirac distribution $\delta : \mathscr{S}(\mathbb{R}) \to \mathbb{R}$, defined by $\delta(\phi) = \phi(0)$ for all $\phi \in \mathscr{S}(\mathbb{R})$.

(10) **c.** Use Eq. $(*)$ to show that $\lim_{a\downarrow 0} T_{f_a} = \delta$, by showing that $\lim_{a\downarrow 0} T_{f_a}(\phi) = \phi(0)$ for all $\phi \in \mathscr{S}(\mathbb{R})$. (*Hint:* Use b, and argue why you may ignore the Lagrange remainder in this limit.)

Substituting Eq. $(*)$ into $T_{f_a}(\phi) = \frac{1}{a}\int_{-a}^{a}\phi(x)\,dx + \frac{1}{a^2}\int_{-a}^{0} x\,\phi(x)\,dx - \frac{1}{a^2}\int_{0}^{a} x\,\phi(x)\,dx$ yields $T_{f_a}(\phi) = \mathrm{I}_0 + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3$, in which, respectively,

$$\begin{aligned}
\mathrm{I}_0 &= \frac{\phi(0)}{a}\int_{-a}^{a} dx = 2\,\phi(0) \\
\mathrm{I}_1 &= \frac{1}{a^2}\int_{-a}^{0} x\,(\phi(0) + \phi'(0)\,x)\,dx = \frac{\phi(0)}{a^2}\int_{-a}^{0} x\,dx + \mathcal{O}(a) = -\frac{1}{2}\phi(0) + \mathcal{O}(a) \\
\mathrm{I}_2 &= -\frac{1}{a^2}\int_{0}^{a} x\,(\phi(0) + \phi'(0)\,x)\,dx = -\frac{\phi(0)}{a^2}\int_{0}^{a} x\,dx + \mathcal{O}(a) = -\frac{1}{2}\phi(0) + \mathcal{O}(a) \\
\mathrm{I}_3 &= \frac{1}{a}\int_{-a}^{a}\mathcal{O}(x^2)\,dx + \frac{1}{a^2}\int_{-a}^{0}\mathcal{O}(x^3)\,dx - \frac{1}{a^2}\int_{0}^{a}\mathcal{O}(x^3)\,dx = \mathcal{O}(a^2)\,.
\end{aligned}$$

Conclusion: $\lim_{a\downarrow 0} T_{f_a}(\phi) = \lim_{a\downarrow 0}(\phi(0) + \mathcal{O}(a)) = \phi(0) = \delta(\phi)$ for all $\phi \in \mathscr{S}(\mathbb{R})$, i.e. $\lim_{a\downarrow 0} T_{f_a} = \delta$.

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(30) **4.** FOURIER TRANSFORMATION

The Fourier convention used in this problem for functions of one variable is as follows:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx \quad \text{whence} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \widehat{f}(\omega)\, d\omega\,.$$

We indicate the Fourier transform of a function f by $\mathscr{F}(f)$, and the inverse Fourier transform of a function $\widehat{f}$ by $\mathscr{F}^{-1}(\widehat{f})$.

You may use the following standard limit, in which $z \in \mathbb{C}$ with real part $\operatorname{Re} z \in \mathbb{R}$:

$$\lim_{\operatorname{Re} z \to -\infty} e^z = 0\,.$$

(5) **a.** Let $\widehat{f}^+$ and $\widehat{f}^-$ be any pair of $\mathbb{C}$-valued functions defined in Fourier space, such that $\widehat{f}^-(\omega) = \widehat{f}^+(-\omega)$. Assuming that the Fourier inverses $f^\pm = \mathscr{F}^{-1}(\widehat{f}^\pm)$ exist, show that $f^-(x) = f^+(-x)$.

$f^-(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, \widehat{f}^-(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, \widehat{f}^+(-\omega)\, d\omega \stackrel{*}{=} -\frac{1}{2\pi}\int_{\infty}^{-\infty} e^{-i\omega' x}\, \widehat{f}^+(\omega')\, d\omega' = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\, \widehat{f}^+(\omega)\, d\omega = f^+(-x)$. In $*$ a new variable $\omega' = -\omega$ has been introduced, all other equalities follow from the given definitions.

We now consider the following particular instances:

$$\widehat{f_s}^+(\omega) = \begin{cases} e^{-s\omega} & \text{if } \omega > 0 \\ \frac{1}{2} & \text{if } \omega = 0 \\ 0 & \text{if } \omega < 0 \end{cases} \qquad (\star)$$

and $\widehat{f_s}^-(\omega) = \widehat{f_s}^+(-\omega)$, in which $s > 0$ is a parameter.

(5) **b.** Give the explicit definition of $\widehat{f_s}^-(\omega)$ in a form similar to that of $\widehat{f_s}^+(\omega)$ in Eq. $(\star)$.

Replacing all instances of ω in Eq. $(\star)$ by $-\omega$ leads to

$$\widehat{f_s}^-(\omega) = \begin{cases} e^{s\omega} & \text{if } \omega < 0 \\ \frac{1}{2} & \text{if } \omega = 0 \\ 0 & \text{if } \omega > 0 \end{cases}$$

(5) **c1.** Compute $f_s^+(x) = \left(\mathscr{F}^{-1}(\widehat{f_s}^+)\right)(x)$.

We have $f_s^+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, \widehat{f_s}^+(\omega)\, d\omega = \frac{1}{2\pi}\int_0^{\infty} e^{\omega(ix-s)}\, d\omega = \frac{1}{2\pi} \frac{e^{\omega(ix-s)}}{ix-s}\Big|_0^{\infty} = \frac{1}{2\pi}\frac{1}{s-ix}$. In the last step we have used the standard limit for the complex exponential function stated above.

(5) **c2.** Compute $f_s^-(x) = \left(\mathscr{F}^{-1}(\widehat{f_s}^-)\right)(x)$.

According to the result under a1 we have $f_s^-(x) = f_s^+(-x) = \frac{1}{2\pi}\frac{1}{s+ix}$.

(5) **d.** We define $\widehat{f_s} = \widehat{f_s}^+ + \widehat{f_s}^-$. Give the explicit form of $\widehat{f_s}(\omega)$ and compute $f_s(x) = \left(\mathscr{F}^{-1}(\widehat{f_s})\right)(x)$.

Since $\mathscr{F}^{-1}$ is a linear operator we have $f_s = \mathscr{F}^{-1}(\widehat{f_s}) = \mathscr{F}^{-1}(\widehat{f_s}^+ + \widehat{f_s}^-) = \mathscr{F}^{-1}(\widehat{f_s}^+) + \mathscr{F}^{-1}(\widehat{f_s}^-) = f_s^+ + f_s^-$. That is, $f_s(x) = \frac{1}{2\pi}\frac{1}{s-ix} + \frac{1}{2\pi}\frac{1}{s+ix} = \frac{1}{\pi}\frac{s}{x^2+s^2}$.

(5) **e.** Show that $\mathscr{F}(f_s * f_t) = \widehat{f}_{s+t}$.

We have $\mathscr{F}(f_s * f_t) \stackrel{*}{=} \mathscr{F}(f_s)\,\mathscr{F}(f_t) = \widehat{f_s}\,\widehat{f_t} \stackrel{\star}{=} \widehat{f}_{s+t}$. In $*$ we have used a well-known Fourier theorem, whereas $\star$ makes explicit use of the property $\widehat{f_s}(\omega) = e^{-s|\omega|}$.

THE END