EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Monday January 17, 2011. Time: 09h00–12h00. Place: AUD 4

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, or other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.

GOOD LUCK!

(**20**) **1.** VECTOR SPACE

We introduce the set $V = \mathbb{R}^2$ and furnish it with an addition and scalar multiplication operator, as follows. For all $(x, y) \in \mathbb{R}^2$, $(u, v) \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$ we define

$$(x, y) + (u, v) = (x + u, y + v) \quad \text{and} \quad \lambda \cdot (x, y) = (\lambda x, y) .$$

Show that, given these definitions, V does *not* constitute a vector space.

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(**25**) **2.** GROUP THEORY[1]

We define the following grey-value transformation: $T_\gamma : \mathbb{R} \to \mathbb{R} : s \mapsto T_\gamma(s) \stackrel{\text{def}}{=} e^{\gamma s}$, in which $\gamma \in \mathbb{R}$ is an arbitrary constant. We furnish the set of all transformations of this type, $G = \{T_\gamma \mid \gamma \in \mathbb{R}\}$, with an infix multiplication operator $\times$, as follows:

$$(T_\alpha \times T_\beta)(s) \stackrel{\text{def}}{=} T_\alpha(s)\, T_\beta(s) \quad \text{for all } s \in \mathbb{R}.$$

a. Prove that G constitutes a group. Proceed as follows:

(5) **a1.** Prove that G is closed with respect to multiplication, i.e. prove that $T_\alpha, T_\beta \in G$ implies $T_\alpha \times T_\beta \in G$ for all $\alpha, \beta \in \mathbb{R}$.

(5) **a2.** Prove that multiplication is associative on G, i.e. prove that $(T_\alpha \times T_\beta) \times T_\gamma = T_\alpha \times (T_\beta \times T_\gamma)$ for all $\alpha, \beta, \gamma \in \mathbb{R}$.

(5) **a3.** Prove that G has a unit element, i.e. that there exists a $\nu \in \mathbb{R}$ such that $T_\nu \times T_\gamma =$

[1] Exam June 28, 2006, problem 3.

$T_\gamma \times T_\nu = T_\gamma$ for all $\gamma \in \mathbb{R}$. Moreover, give the explicit value of $\nu \in \mathbb{R}$ corresponding to this unit element $T_\nu \in G$.

(5) **a4.** Finally prove that each element of G has an inverse, i.e. that for each $\eta \in \mathbb{R}$ there exists a $\theta \in \mathbb{R}$ such that $T_\eta \times T_\theta = T_\theta \times T_\eta = T_\nu$, in which $\nu \in \mathbb{R}$ denotes the parameter value corresponding to the unit element in part a3.

(5) **b.** Is G commutative? If yes, prove, if no, provide a counterexample.

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(**25**) **3.** Distribution Theory

The parameterized function $f_a : \mathbb{R} \to \mathbb{R}$, with parameter $a > 0$, is defined as follows:

$$f_a(x) = \begin{cases} \dfrac{1}{a^2}(-|x| + a) & \text{if } x \in [-a, a] \\ 0 & \text{elsewhere} \end{cases}$$

(5) **a.** Sketch the graph of $y = f_a(x)$ in the xy-plane, and compute the area enclosed by this graph and the x-axis.

The regular tempered distribution $T_{f_a} : \mathscr{S}(\mathbb{R}) \to \mathbb{R}$ associated with the function f_a is given by

$$T_{f_a}(\phi) = \int_{-\infty}^{\infty} f_a(x)\, \phi(x)\, dx$$

for any smooth test function $\phi \in \mathscr{S}(\mathbb{R})$.

(10) **b.** Show that $T_{f_a}(\phi) = \dfrac{1}{a} \displaystyle\int_{-a}^{a} \phi(x)\, dx + \dfrac{1}{a^2} \int_{-a}^{0} x\, \phi(x)\, dx - \dfrac{1}{a^2} \int_{0}^{a} x\, \phi(x)\, dx.$

We now consider the limit of vanishing parameter $a \downarrow 0$. It is clear that the function f_a is ill-defined in this limit. We wish to investigate whether the regular tempered distribution T_{f_a} does have a well-defined limit. To this end we recall Taylor's theorem, which allows us to use the following second order expansion for the test function around the origin:

$$\phi(x) = \phi(0) + \phi'(0)\, x + \frac{1}{2}\phi''(\xi(x))\, x^2\,, \qquad (*)$$

for any $x \in (-a, a)$ and some $\xi(x)$ in-between x and 0. The last term on the right hand side is referred to as the Lagrange remainder, and is sometimes simplified as $\mathcal{O}(x^2)$.

Finally, recall the Dirac distribution $\delta : \mathscr{S}(\mathbb{R}) \to \mathbb{R}$, defined by $\delta(\phi) = \phi(0)$ for all $\phi \in \mathscr{S}(\mathbb{R})$.

(10) **c.** Use Eq. $(*)$ to show that $\lim\limits_{a \downarrow 0} T_{f_a} = \delta$, by showing that $\lim\limits_{a \downarrow 0} T_{f_a}(\phi) = \phi(0)$ for all $\phi \in \mathscr{S}(\mathbb{R})$.
(*Hint:* Use b, and argue why you may ignore the Lagrange remainder in this limit.)

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(**30**) **4.** Fourier Transformation

The Fourier convention used in this problem for functions of one variable is as follows:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\, f(x)\, dx \quad \text{whence} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, \widehat{f}(\omega)\, d\omega\,.$$

We indicate the Fourier transform of a function f by $\mathscr{F}(f)$, and the inverse Fourier transform of a function $\widehat{f}$ by $\mathscr{F}^{-1}(\widehat{f})$.

You may use the following standard limit, in which $z \in \mathbb{C}$ with real part $\operatorname{Re} z \in \mathbb{R}$:

$$\lim_{\operatorname{Re} z \to -\infty} e^{z} = 0\,.$$

(5) **a.** Let $\widehat{f}^{+}$ and $\widehat{f}^{-}$ be any pair of $\mathbb{C}$-valued functions defined in Fourier space, such that $\widehat{f}^{-}(\omega) = \widehat{f}^{+}(-\omega)$. Assuming that the Fourier inverses $f^{\pm} = \mathscr{F}^{-1}(\widehat{f}^{\pm})$ exist, show that $f^{-}(x) = f^{+}(-x)$.

We now consider the following particular instances:

$$\widehat{f}_s^{+}(\omega) = \begin{cases} e^{-s\omega} & \text{if } \omega > 0 \\ \frac{1}{2} & \text{if } \omega = 0 \\ 0 & \text{if } \omega < 0 \end{cases} \qquad (\star)$$

and $\widehat{f}_s^{-}(\omega) = \widehat{f}_s^{+}(-\omega)$, in which $s > 0$ is a parameter.

(5) **b.** Give the explicit definition of $\widehat{f}_s^{-}(\omega)$ in a form similar to that of $\widehat{f}_s^{+}(\omega)$ in Eq. $(\star)$.

(5) **c1.** Compute $f_s^{+}(x) = \left(\mathscr{F}^{-1}(\widehat{f}_s^{+})\right)(x)$.

(5) **c2.** Compute $f_s^{-}(x) = \left(\mathscr{F}^{-1}(\widehat{f}_s^{-})\right)(x)$.

(5) **d.** We define $\widehat{f}_s = \widehat{f}_s^{+} + \widehat{f}_s^{-}$. Give the explicit form of $\widehat{f}_s(\omega)$ and compute $f_s(x) = \left(\mathscr{F}^{-1}(\widehat{f}_s)\right)(x)$.

(5) **e.** Show that $\mathscr{F}(f_s * f_t) = \widehat{f}_{s+t}$.

THE END